

On quasi-categories of comodules and Landweber exactness

Takeshi Torii

Abstract In this paper we study quasi-categories of comodules over coalgebras in a stable homotopy theory. We show that the quasi-category of comodules over the coalgebra associated to a Landweber exact \mathbb{S} -algebra depends only on the height of the associated formal group. We also show that the quasi-category of $E(n)$ -local spectra is equivalent to the quasi-category of comodules over the coalgebra $A \otimes A$ for any Landweber exact $\mathbb{S}_{(p)}$ -algebra of height n at a prime p . Furthermore, we show that the category of module objects over a discrete model of the Morava E -theory spectrum in the $K(n)$ -local discrete symmetric \mathbb{G}_n -spectra is a model of the $K(n)$ -local category, where \mathbb{G}_n is the extended Morava stabilizer group.

1 Introduction

It is known that the stable homotopy category of spectra is intimately related to the theory of formal groups through complex cobordism and Adams-Novikov spectral sequence by the works of Morava [27], Miller-Ravenel-Wilson [26], Devinatz-Hopkins-Smith [8], Hopkins-Smith [11], Hovey-Strickland [16] and many others. The E_2 -page of the Adams-Novikov spectral sequence is described as the derived functor of taking primitives in the abelian category of graded comodules over the co-operation Hopf algebroid associated to the complex cobordism spectrum.

We also have a localized version of Adams-Novikov spectral sequence. For example, for a Landweber exact spectrum E of height n at a prime p , we have an E -based Adams-Novikov spectral sequence abutting to the homotopy groups of E -local spectra. In this case the E -localization and the E_2 -page of the E -based Adams-Novikov spectral sequence depends only on the height n of the associated formal group at p . There are many results that the derived functor describing the E_2 -page

Takeshi Torii
Department of Mathematics, Okayama University, Okayama 700–8530, Japan,
e-mail: torii@math.okayama-u.ac.jp

of the E -based Adams-Novikov spectral sequence depends only on the substack of the moduli stack of formal groups [13], [14], [17], [28].

These results suggest that there may be an intimate relationship between localized quasi-categories of spectra and quasi-categories of comodules over co-operation coalgebras. In this paper we investigate this relationship. We show that the quasi-category of comodules over a coalgebra associated to a Landweber exact \mathbb{S} -algebra depends only on the height of the associated formal group and that the quasi-category of comodules over a coalgebra associated to a Landweber exact $\mathbb{S}_{(p)}$ -algebra of height n at a prime p is equivalent to the quasi-category of $E(n)$ -local spectra, where $E(n)$ is the n th Johnson-Wilson spectrum at p .

First, we introduce a quasi-category of comodules over a coalgebra associated to an algebra object of a stable homotopy theory \mathcal{C} . In this paper we regard coalgebra objects as algebra objects of the opposite monoidal quasi-category of A - A -bimodule objects for an algebra object A of \mathcal{C} . We regard comodule objects over a coalgebra Γ as module objects over Γ in the opposite quasi-category of A -module objects in \mathcal{C} . In particular, we show that $A \otimes A$ is a coalgebra object for an algebra object A of \mathcal{C} and we can consider the quasi-category

$$\mathrm{LComod}_{\Gamma(A)}(\mathcal{C})$$

of left comodules over $A \otimes A$ in \mathcal{C} , where $\Gamma(A)$ represents the pair $(A, A \otimes A)$. For a map $A \rightarrow B$ of algebra objects of \mathcal{C} , we have the extension of scalars functor $B \otimes_A (-) : \mathrm{LMod}_A(\mathcal{C}) \rightarrow \mathrm{LMod}_B(\mathcal{C})$, where $\mathrm{LMod}_A(\mathcal{C})$ and $\mathrm{LMod}_B(\mathcal{C})$ are the quasi-categories of left A -modules and B -modules, respectively. We show that the extension of scalars functor extends to a functor

$$B \otimes_A (-) : \mathrm{LComod}_{\Gamma(A)}(\mathcal{C}) \longrightarrow \mathrm{LComod}_{\Gamma(B)}(\mathcal{C}).$$

of quasi-categories of comodules.

Next, we consider Landweber exact \mathbb{S} -algebras in the quasi-category of spectra Sp , where \mathbb{S} is the sphere spectrum. We show that, if A is a Landweber exact \mathbb{S} -algebra, then the quasi-category of comodules over the coalgebra $A \otimes A$ depends only on the height of the associated formal group.

Theorem 1 (cf. Theorem 8). *If A and B are Landweber exact \mathbb{S} -algebras with the same height at all primes p , then there is an equivalence of quasi-categories*

$$\mathrm{LComod}_{\Gamma(A)}(\mathrm{Sp}) \simeq \mathrm{LComod}_{\Gamma(B)}(\mathrm{Sp}).$$

We also show that the quasi-category of comodules over $A \otimes A$ is equivalent to the quasi-category $L_n \mathrm{Sp}$ of $E(n)$ -local spectra if A is a Landweber exact $\mathbb{S}_{(p)}$ -algebra of height n at a prime p .

Theorem 2 (cf. Theorem 9). *If A is a Landweber exact $\mathbb{S}_{(p)}$ -algebra of height n at a prime p , then there is an equivalence of quasi-categories*

$$L_n \mathrm{Sp} \simeq \mathrm{LComod}_{\Gamma(A)}(\mathrm{Sp}).$$

As an application of the results in this paper we show that the model category constructed in [31] is a model of the $K(n)$ -local category, where $K(n)$ is the n th Morava K -theory spectrum at a prime p . We denote by $\Sigma\mathrm{Sp}$ the model category of symmetric spectra and by $\Sigma\mathrm{Sp}_{K(n)}$ its left Bousfield localization of $\Sigma\mathrm{Sp}$ with respect to $K(n)$. The n th extended Morava stabilizer group \mathbb{G}_n is a profinite group and we can consider the model category $\Sigma\mathrm{Sp}(\mathbb{G}_n)$ of discrete symmetric \mathbb{G}_n -spectra and its Bousfield localization $\Sigma\mathrm{Sp}(\mathbb{G}_n)_{K(n)}$ with respect to $K(n)$. We have a commutative monoid object F_n in $\Sigma\mathrm{Sp}(\mathbb{G}_n)_{K(n)}$ constructed by Davis [7] and Behrens-Davis [5], which is a discrete model of the n th Morava E -theory spectrum E_n . In [31] we showed that the extension of scalars functor

$$L_{K(n)}(F_n \otimes (-)) : \Sigma\mathrm{Sp}(\mathbb{G}_n)_{K(n)} \longrightarrow \mathrm{LMod}_{F_n}(\Sigma\mathrm{Sp}(\mathbb{G}_n)_{K(n)}),$$

which is a left Quillen functor, is homotopically fully faithful, that is, it induces a weak homotopy equivalence between mapping spaces for any two objects in $\Sigma\mathrm{Sp}(\mathbb{G}_n)_{K(n)}$. In this paper we show that this functor is actually a left Quillen equivalence and hence we can consider the category $\mathrm{LMod}_{F_n}(\Sigma\mathrm{Sp}(\mathbb{G}_n)_{K(n)})$ is a model of the $K(n)$ -local category.

Theorem 3 (cf. Theorem 11). *The extension of scalars functor*

$$L_{K(n)}(F_n \otimes (-)) : \Sigma\mathrm{Sp}(\mathbb{G}_n)_{K(n)} \longrightarrow \mathrm{LMod}_{F_n}(\Sigma\mathrm{Sp}(\mathbb{G}_n)_{K(n)})$$

is a left Quillen equivalence.

The organization of this paper is as follows: In §2 we fix some notation we use throughout this paper. In §3 we study opposite coCartesian fibrations, opposite monoidal quasi-categories, and opposite tensored quasi-categories. In particular, we show that a lax monoidal right adjoint functor between monoidal quasi-categories induces a lax monoidal right adjoint functor between the opposite monoidal quasi-categories. In §4 we introduce a quasi-category of comodules over a coalgebra in a stable homotopy theory. We define a cotensor product of a right comodule and a left comodule over a coalgebra as a limit of the cobar construction. We study the relationship between localizations of a stable homotopy theory and quasi-categories of comodules. In §5 we study comodules in spectra over a coalgebra associated to a Landweber exact \mathbb{S} -algebra. First, we study the Bousfield-Kan spectral sequence associated to the two-sided cobar construction. Next, we show that the quasi-category of comodules over the coalgebra associated to a Landweber exact \mathbb{S} -algebra depends only on the height of the associated formal group. Finally, we show that the model category of modules over F_n in the $K(n)$ -local discrete symmetric \mathbb{G}_n -spectra is a model of the $K(n)$ -local category. In §6 we give a proof of Proposition 1 stated in §3, which is technical but important for constructing a canonical map between opposite coCartesian fibrations.

2 Notation

For a category \mathcal{C} , we denote by $\mathrm{Hom}_{\mathcal{C}}(x, y)$ the set of all morphisms from x to y in \mathcal{C} for $x, y \in \mathcal{C}$.

We denote by \mathbf{sSet} the category of simplicial sets. For a simplicial set K , we denote by K^{op} the opposite simplicial set (see [20, §1.2.1]). If K is a quasi-category, then K^{op} is also a quasi-category. For simplicial sets X, Y , we denote by $\mathrm{Fun}(X, Y)$ the simplicial mapping space from X to Y . For a simplicial set X equipped with a map $\pi : X \rightarrow S$ of simplicial sets, we denote by X_s the fiber of π over $s \in S$. If X and Y are simplicial sets over a simplicial set S , then we denote by $\mathrm{Fun}_S(X, Y)$ the simplicial set of maps from X to Y over S .

For a small (simplicial) category \mathcal{C} , we denote by $N(\mathcal{C})$ the simplicial set obtained by applying the (simplicial) nerve functor $N(-)$ to \mathcal{C} (see [20, §1.1.5]). We denote by Cat_{∞} the quasi-category of (small) quasi-categories (see [20, §3]).

We denote by $\Sigma\mathrm{Sp}$ the category of symmetric spectra equipped with the stable model structure (see [15]). We denote by Sp the quasi-category of spectra, which is the underlying quasi-category of the simplicial model category $\Sigma\mathrm{Sp}$. We denote by $\mathrm{Ho}(\mathrm{Sp})$ the stable homotopy category of spectra. We denote by \mathbb{S} the sphere spectrum. For a spectrum $X \in \mathrm{Sp}$, we write X_* for the homotopy groups π_*X . For spectra $X, Y \in \mathrm{Sp}$, we write $X \otimes Y$ the smash product of X and Y .

3 Opposite monoidal quasi-categories and opposite tensored quasi-categories over monoidal quasi-categories

In this section we study the opposite quasi-categories of monoidal quasi-categories and the opposite quasi-categories of tensored quasi-categories over monoidal quasi-categories. The author thinks that the results in this section are well-known to experts but he decided to include this section because he does not find out appropriate references.

In §3.1 we recall a model of opposite coCartesian fibrations by Barwick-Glasman-Nardin [2] and study maps between opposite coCartesian fibrations. In §3.2 we study the opposite quasi-category of a monoidal quasi-category and show that a lax monoidal right adjoint functor between monoidal quasi-categories induces a lax monoidal right adjoint functor between the opposite monoidal quasi-categories. In §3.3 we study the opposite of a tensored quasi-category over a monoidal quasi-category. We show that a lax tensored right adjoint functor between tensored quasi-categories induces a lax tensored right adjoint functor between the opposites of the tensored quasi-categories.

3.1 Opposite coCartesian fibrations

For a coCartesian fibration we have the opposite coCartesian fibration whose fibers are the opposite quasi-categories of the fibers of the original coCartesian fibration. In this subsection we recall the explicit model of opposite coCartesian fibrations due to Barwick-Glasman-Nardin [2]. We show that a map between coCartesian fibrations whose restriction to every fiber admits a left adjoint induces a map between the opposite coCartesian fibrations.

First, we recall the explicit model of opposite coCartesian fibrations by Barwick-Glasman-Nardin [2].

Let S be a simplicial set and let $p : X \rightarrow S$ be a coCartesian fibration with small fibers. We denote by X_s the quasi-category that is the fiber of p over $s \in S$. Let Cat_∞ be the quasi-category of small quasi-categories. By [20, §3.3.2], the coCartesian fibration p is classified by a functor $\mathbf{X} : S \rightarrow \text{Cat}_\infty$. There is an involution

$$R : \text{Cat}_\infty \longrightarrow \text{Cat}_\infty$$

carrying a quasi-category to its opposite. The composite functor $R\mathbf{X}$ classifies a coCartesian fibration $Rp : RX \rightarrow S$ in which the fiber $(RX)_s$ of Rp over $s \in S$ is equivalent to the opposite quasi-category $(X_s)^{\text{op}}$ for all $s \in S$. We call $Rp : RX \rightarrow S$ the opposite coCartesian fibration of $p : X \rightarrow S$. In the following of this subsection we assume that the base simplicial set S is a quasi-category.

To describe the model of opposite coCartesian fibrations, we recall the twisted arrow quasi-category. The twisted arrow quasi-category $\tilde{\mathcal{O}}(K)$ for a quasi-category K is the simplicial set in which the set of n -simplexes is given by

$$\tilde{\mathcal{O}}(K)_n = \text{Hom}_{\text{Set}_\Delta}((\Delta^n)^{\text{op}} \star \Delta^n, K)$$

with obvious structure maps. The simplicial set $\tilde{\mathcal{O}}(K)$ is actually a quasi-category (see [22, Prop. 4.2.3]). Note that the inclusions $\Delta^n \hookrightarrow (\Delta^n)^{\text{op}} \star \Delta^n$ and $(\Delta^n)^{\text{op}} \hookrightarrow (\Delta^n)^{\text{op}} \star \Delta^n$ induce maps of simplicial sets $\tilde{\mathcal{O}}(K) \rightarrow K$ and $\tilde{\mathcal{O}}(K) \rightarrow K^{\text{op}}$, respectively.

We use the twisted arrow category $\tilde{\mathcal{O}}(\Delta^n)$ for the n -simple Δ^n for $n \geq 0$ to describe the model of opposite coCartesian fibrations. The twisted arrow quasi-category $\tilde{\mathcal{O}}(\Delta^n)$ for Δ^n is the nerve of $\widetilde{[n]}$, where $\widetilde{[n]}$ is the ordered set of all pairs (i, j) of integers with $0 \leq i \leq j \leq n$ equipped with order relation $(i, j) \leq (i', j')$ if and only if $i \geq i'$ and $j \leq j'$. The ordered set $\widetilde{[n]}$ is depicted as follows

$$\begin{array}{ccccccc}
00 & \rightarrow & 01 & \rightarrow & 02 & \rightarrow & \cdots & \rightarrow & 0\bar{1} & \rightarrow & 0\bar{0} \\
& & \uparrow & & \uparrow & & & & \uparrow & & \uparrow \\
& & 11 & \rightarrow & 12 & \rightarrow & \cdots & \rightarrow & 1\bar{1} & \rightarrow & 1\bar{0} \\
& & & & \uparrow & & & & \uparrow & & \uparrow \\
& & & & 22 & \rightarrow & \cdots & \rightarrow & 2\bar{1} & \rightarrow & 2\bar{0} \\
& & & & & & & & \uparrow & & \uparrow \\
& & & & & & \ddots & & \vdots & & \vdots \\
& & & & & & & & \uparrow & & \uparrow \\
& & & & & & & & \bar{1}1 & \rightarrow & \bar{1}0 \\
& & & & & & & & & & \uparrow \\
& & & & & & & & & & \bar{0}0,
\end{array} \tag{1}$$

where $\bar{k} = n - k$.

By functoriality of $\tilde{\mathcal{O}}(-)$, we have a cosimplicial simplicial set $\tilde{\mathcal{O}}(\Delta^\bullet)$. For a simplicial set K over S , we define a simplicial set $H(K)$ over S as follows. The simplicial set $H(K)$ is a simplicial subset of $\text{Hom}_{\text{sSet}}(\tilde{\mathcal{O}}(\Delta^\bullet), K)$. A map $\varphi : \tilde{\mathcal{O}}(\Delta^n) \rightarrow K$ is an n -simplex of $H(K)$ for $n \geq 0$ if the j -simplex $\varphi(jj) \rightarrow \cdots \rightarrow \varphi(1j) \rightarrow \varphi(0j)$ covers a totally degenerate j -simplex of S , that is, a j -simplex in the image of the map $S_0 \rightarrow S_j$, for all $0 \leq j \leq n$. Assigning to an n -simplex φ of $H(K)$ the n -simplex $p\varphi(00) \rightarrow p\varphi(01) \rightarrow \cdots \rightarrow p\varphi(0n)$ of S , we obtain a map $H(K) \rightarrow S$.

Let $p : X \rightarrow S$ be a coCartesian fibration, where S is a quasi-category. We define a simplicial set RX as follows. The simplicial set RX is a simplicial subset of $H(X)$. A map $\varphi : \tilde{\mathcal{O}}(\Delta^n) \rightarrow X$ is an n -simplex of RX for $n \geq 0$ if the following two conditions are satisfied:

1. The j -simplex $\varphi(jj) \rightarrow \cdots \rightarrow \varphi(1j) \rightarrow \varphi(0j)$ covers a totally degenerate j -simplex of S for all $0 \leq j \leq n$.
2. The 1-simplex $\varphi(ij) \rightarrow \varphi(ik)$ is a p -coCartesian edge for all $0 \leq i \leq j \leq k \leq n$.

As in $H(X)$, we have a map

$$Rp : RX \rightarrow S,$$

which is a coCartesian fibration. The fiber $(RX)_s$ over $s \in S$ is equivalent to the opposite quasi-category $(X_s)^{\text{op}}$ of the fiber X_s for all $s \in S$. An edge $\varphi \in \text{Hom}_{\text{sSet}}(\Delta^1, RX)$ is Rp -coCartesian if and only if the edge $\varphi(11) \rightarrow \varphi(01)$ is an equivalence in the fiber X_s , where $s = p\varphi(11)$. The coCartesian fibration $Rp : RX \rightarrow S$ is a model of the opposite coCartesian fibration corresponding to the composite

$$RX : S \xrightarrow{\mathbf{X}} \text{Cat}_\infty \xrightarrow{R} \text{Cat}_\infty,$$

where $\mathbf{X} : S \rightarrow \text{Cat}_\infty$ is the map corresponding to the coCartesian fibration $p : X \rightarrow S$, and $R : \text{Cat}_\infty \rightarrow \text{Cat}_\infty$ is the functor which assigns to a quasi-category its opposite quasi-category.

Next, we consider a map between coCartesian fibrations which admits a left adjoint for each fibers. We show that the map induces a canonical map in the opposite direction between the opposite coCartesian fibrations.

Let $p : X \rightarrow S$ and $q : Y \rightarrow S$ be coCartesian fibrations over a quasi-category S . Suppose we have a map $G : Y \rightarrow X$ over S . Note that we do not assume that G preserves coCartesian edges. The map $G : Y \rightarrow X$ over S induces a functor $G_s : Y_s \rightarrow X_s$ between the quasi-categories of fibers for each $s \in S$.

We shall define a simplicial set \mathcal{R} over S equipped with maps $\pi_X : \mathcal{R} \rightarrow RX$ and $\pi_Y : \mathcal{R} \rightarrow RY$ over S . For a simplicial set K and X , we denote by $\text{Fun}(K, X)$ the mapping simplicial set from K to X . The map $p : X \rightarrow S$ induces a map $p_* : \text{Fun}(\Delta^1, X) \rightarrow \text{Fun}(\Delta^1, S)$. We regard S as a simplicial subset of $\text{Fun}(\Delta^1, S)$ via constant maps. We denote by $\text{Fun}^S(\Delta^1, X)$ the pullback of p_* along the inclusion $S \hookrightarrow \text{Fun}(\Delta^1, S)$. The inclusion $\Delta^{\{i\}} \hookrightarrow \Delta^1$ induces a map $\text{Fun}^S(\Delta^1, X) \rightarrow X$ over S for $i = 0, 1$.

We have the inclusion $RX \hookrightarrow H(X) \cong H(\text{Fun}(\Delta^{\{0\}}, X))$. The map $G : Y \rightarrow X$ over S induces a map $RY \hookrightarrow H(Y) \xrightarrow{G_*} H(X) \cong H(\text{Fun}(\Delta^{\{1\}}, X))$. The inclusion $\Delta^{\{i\}} \hookrightarrow \Delta^1$ induces a map $H(\text{Fun}^S(\Delta^1, X)) \rightarrow H(\text{Fun}(\Delta^{\{i\}}, X))$ for $i = 0, 1$. Using these maps, we define a simplicial set \mathcal{R} by

$$\mathcal{R} = RX \times_{H(\text{Fun}(\Delta^{\{0\}}, X))} H(\text{Fun}^S(\Delta^1, X)) \times_{H(\text{Fun}(\Delta^{\{1\}}, X))} RY.$$

We have a map $\mathcal{R} \rightarrow S$ and projections $\pi_X : \mathcal{R} \rightarrow RX$ and $\pi_Y : \mathcal{R} \rightarrow RY$ over S .

Now we assume that the functor $G_s : Y_s \rightarrow X_s$ admits a left adjoint F_s for all $s \in S$. Then an object x of X with $s = p(x)$ determines an object $(x, u_x, F_s(x))$ of \mathcal{R} , where $u_x : x \rightarrow G_s F_s(x)$ is the unit map of the adjunction (F_s, G_s) at x . We define \mathcal{R}^0 to be the full subcategory of \mathcal{R} spanned by $\{(x, u_x, F_s(x))\}$ for all $x \in X$, where $s = p(x)$. Let

$$\pi_X^0 : \mathcal{R}^0 \longrightarrow RX.$$

be the restriction of π_X to \mathcal{R}^0 .

Proposition 1. *The map $\pi_X^0 : \mathcal{R}^0 \rightarrow RX$ is a trivial Kan fibration.*

We defer the proof of Proposition 1 to §6.

We take a section T_0 of π_X^0 , which is unique up to contractible space of choices. Let $\pi_Y^0 : \mathcal{R}^0 \rightarrow RY$ be the restriction of π_Y to \mathcal{R}^0 . We define a functor

$$RF : RX \longrightarrow RY$$

to be $\pi_Y^0 T_0$.

We would like to describe some properties of the section T_0 . Let $s \in S$. We consider the restriction of T_0 to $(RX)_s$. The fiber \mathcal{R}_s is described as

$$\mathcal{R}_s = (RX)_s \times_{H(\text{Fun}(\Delta^{\{0\}}, X_s))} H(\text{Fun}(\Delta^1, X_s)) \times_{H(\text{Fun}(\Delta^{\{1\}}, X_s))} (RY)_s,$$

and the fiber \mathcal{R}_s^0 is a full subcategory of \mathcal{R}_s . The composition $(RX)_s \hookrightarrow H(X_s) \xrightarrow{(F_s)_*} H(Y_s)$ factors through $(RY)_s$. We denote by $R(F_s)$ the induced functor $(RX)_s \rightarrow$

$(RY)_s$. The unit map $u_s : 1_{X_s} \rightarrow G_s F_s$ in $\text{Fun}(X_s, X_s)$ can be identified with a map $u_s : X_s \rightarrow \text{Fun}(\Delta^1, X_s)$. We obtain a map $Hu_s : (RX)_s \rightarrow H(\text{Fun}(\Delta^1, X_s))$ by the composition $(RX)_s \hookrightarrow H(X_s) \xrightarrow{(u_s)^*} H(\text{Fun}(\Delta^1, X_s))$. Note that Hu_s followed by $H(\text{Fun}(\Delta^1, X_s)) \rightarrow H(\text{Fun}(\Delta^{\{0\}}, X_s))$ is the inclusion $(RX)_s \hookrightarrow H(X_s)$, and the map Hu_s followed by $H(\text{Fun}(\Delta^1, X_s)) \rightarrow H(\text{Fun}(\Delta^{\{1\}}, X_s))$ is the composition of $R(F_s) : (RX)_s \rightarrow (RY)_s$ followed by the inclusion $(RY)_s \hookrightarrow H(X_s)$. Hence we obtain a section of \mathcal{R}_s^0 over $(RX)_s$:

$$(1_{(RX)_s}, Hu_s, R(F_s)) : (RX)_s \longrightarrow \mathcal{R}_s^0.$$

Proposition 2. *We have*

$$T_0|_{(RX)_s} \simeq (1_{(RX)_s}, Hu_s, R(F_s)).$$

for any $s \in S$.

Proof. Restricting π_X^0 to the fibers over $s \in S$, we obtain a trivial Kan fibration $(\pi_X^0)_s : \mathcal{R}_s^0 \rightarrow (RX)_s$. The restriction of the section T_0 to $(RX)_s$ is a section of $(\pi_X^0)_s$. The map $(1_{(RX)_s}, Hu_s, R(F_s))$ is also a section of $(\pi_X^0)_s$. Hence we have $T_0|_{(RX)_s} \simeq (1_{(RX)_s}, Hu_s, R(F_s))$. \square

Next, we consider the image of edges of RX under the section T_0 . Let φ be an edge of RX over $e : s \rightarrow s'$ in S represented by a p -coCartesian edge $\varphi(00) \rightarrow \varphi(01)$ in X and an edge $\varphi(11) \rightarrow \varphi(01)$ in the fiber $X_{s'}$. We take a q -coCartesian edge $\psi : F_s \varphi(00) \rightarrow y'$ in Y over e . Since $\varphi(00) \rightarrow \varphi(01)$ is p -coCartesian, we obtain an edge $\varphi(01) \rightarrow G_{s'} y'$ in $X_{s'}$, which makes the following diagram commute

$$\begin{array}{ccc} \varphi(00) & \longrightarrow & \varphi(01) \\ \downarrow u & & \vdots \\ G_s F_s \varphi(00) & \xrightarrow{G\psi} & G_{s'} y', \end{array} \quad (2)$$

where u is the unit map of the adjunction (F_s, G_s) at $\varphi(00)$. Let $w : F_{s'} \varphi(11) \rightarrow y'$ be the map in $Y_{s'}$ obtained from $\varphi(11) \rightarrow \varphi(01)$ by applying $F_{s'}$ followed by the adjoint map of $\varphi(01) \rightarrow G_{s'} y'$. We denote by $R\varphi$ the edge of RY over e represented by

$$F_s \varphi(00) \xrightarrow{\psi} y' \xleftarrow{w} F_{s'} \varphi(11).$$

Since the composite $\varphi(11) \rightarrow \varphi(01) \rightarrow G_{s'} y'$ is adjoint to $w : F_{s'} \varphi(11) \rightarrow y'$, we have an edge $H\varphi : \varphi \rightarrow G(R\varphi)$ of $H(\text{Fun}^S(\Delta^1, X))$ represented by the following commutative diagram

$$\begin{array}{ccccc} \varphi(00) & \longrightarrow & \varphi(01) & \longleftarrow & \varphi(11) \\ \downarrow u & & \downarrow & & \downarrow u \\ G_s F_s \varphi(00) & \xrightarrow{G\psi} & G_{s'} y' & \xleftarrow{G_{s'} w} & G_{s'} F_{s'} \varphi(11). \end{array}$$

Proposition 3. *For any edge φ of RX , we have*

$$T_0(\varphi) \simeq (\varphi, H\varphi, R\varphi).$$

Proof. Let $\pi_\varphi^0 : \mathcal{R}_\varphi^0 \rightarrow \Delta^1$ be the trivial Kan fibration obtained by the pullback of π_X^0 along the map $\varphi : \Delta^1 \rightarrow RX$. The triple $(\varphi, H\varphi, R\varphi)$ determines a section of π_φ^0 . Hence $T_0(\varphi) \simeq (\varphi, H\varphi, R\varphi)$. \square

The main result in this subsection is the following theorem.

Theorem 4. *Let $p : X \rightarrow S$ and $q : Y \rightarrow S$ be coCartesian fibrations over a quasi-category S . Suppose we have a map $G : Y \rightarrow X$ over S . If G_s admits a left adjoint F_s for all $s \in S$, then there exists a canonical map $RF : RX \rightarrow RY$ over S up to contractible space of choices. We have $(RF)_s \simeq F_s^{\text{op}}$ for all $s \in S$ and $RF(\varphi) \simeq R\varphi$ for any edge φ of RX .*

Proof. The theorem follows from Propositions 1, 2, and 3. \square

Now we consider which coCartesian edge of RX is preserved by the functor RF . Let $e : s \rightarrow s'$ be a 1-simplex of S . Since $q : Y \rightarrow S$ and $p : X \rightarrow S$ are coCartesian fibrations, we have functors $e_!^Y : Y_s \rightarrow Y_{s'}$ and $e_!^X : X_s \rightarrow X_{s'}$ associated to e . The map $G : Y \rightarrow X$ over S induces a diagram $\partial(\Delta^1 \times \Delta^1) \rightarrow \text{Cat}_\infty$ depicted as

$$\begin{array}{ccc} Y_s & \xrightarrow{G_s} & X_s \\ e_!^Y \downarrow & & \downarrow e_!^X \\ Y_{s'} & \xrightarrow{G_{s'}} & X_{s'} \end{array} \quad (3)$$

and a natural transformation

$$e_!^X G_s \longrightarrow G_{s'} e_!^Y. \quad (4)$$

If natural transformation (4) is an equivalence, then $G : Y \rightarrow X$ preserves coCartesian edges over e .

We recall the definition of left adjointable diagram (see [21, Def. 4.7.5.13]). Suppose we are given a diagram of quasi-categories

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{G} & \mathcal{D} \\ U \downarrow & & \downarrow V \\ \mathcal{C}' & \xrightarrow{G'} & \mathcal{D}' \end{array}$$

which commutes up to a specified equivalence $\alpha : VG \simeq G'U$. We say that this diagram is left adjointable if the functors G and G' admit left adjoints F and F' , respectively, and if the composite transformation

$$F'V \rightarrow F'VGF \xrightarrow{\alpha} F'G'UF \rightarrow UF$$

is an equivalence, where the first map is induced by the unit map of the adjunction (F, G) , and the third map is induced by the counit map of the adjunction (F', G') .

Proposition 4. *Let e be a 1-simplex of S . If natural transformation (4) is an equivalence and diagram (3) equipped with this equivalence is left adjointable, then $RF : RX \rightarrow RY$ preserves coCartesian edges over e .*

Proof. Let φ be an Rp -coCartesian edge of RX over e represented by $\varphi(00) \rightarrow \varphi(01) \leftarrow \varphi(11)$, where $\varphi(00) \rightarrow \varphi(01)$ is a p -coCartesian edge of X over e and $\varphi(11) \rightarrow \varphi(01)$ is an equivalence in $X_{s'}$. We can regard $\varphi(11)$ as $e_!^X \varphi(00)$.

Suppose that the edge $R\varphi$ of RY over e is represented by

$$F_s \varphi(00) \xrightarrow{\psi} y' \xleftarrow{w} F_{s'} \varphi(11),$$

where ψ is a q -coCartesian edge of Y over e and w is an edge of $Y_{s'}$. We have to show that w is an equivalence of $Y_{s'}$.

We can regard y' as $e_!^Y F_s \varphi(00)$ and $F_{s'} \varphi(11)$ as $F_{s'} e_!^X \varphi(00)$. The morphism w is the adjoint of the morphism $\varphi(11) \rightarrow \varphi(01) \rightarrow G_{s'} y'$, which can be identified with $e_!^X \varphi(00) \rightarrow G_{s'} e_!^Y F_s \varphi(00)$. By the assumption that diagram (3) is left adjointable, we see that w is an equivalence. \square

3.2 Opposite monoidal quasi-categories

In this subsection we study the opposite monoidal quasi-category of a monoidal quasi-category. We show that a lax right adjoint functor between monoidal quasi-categories induces a lax right adjoint functor between the opposite monoidal quasi-categories.

First, we recall the definition of monoidal quasi-categories. Let $p : M \rightarrow N(\Delta)^{\text{op}}$ be a coCartesian fibration of simplicial sets. For any $n \geq 0$, the inclusion $[1] \cong \{i-1, i\} \hookrightarrow [n]$ induces a functor $p_i : X_{[n]} \rightarrow X_{[1]}$ of quasi-categories for $i = 1, \dots, n$. We say that p is a monoidal quasi-category if the functor

$$p_1 \times \cdots \times p_n : X_{[n]} \longrightarrow \overbrace{X_{[1]} \times \cdots \times X_{[1]}}^n$$

is a categorical equivalence for all $n \geq 0$. The fiber $M_{[1]}$ of p over $[1] \in \Delta$ is said to be the underlying quasi-category of the monoidal category p .

Let $p : M \rightarrow N(\Delta)^{\text{op}}$ be a monoidal quasi-category. Since p is a coCartesian fibration by definition, we have a functor $\mathbf{X} : N(\Delta)^{\text{op}} \rightarrow \text{Cat}_\infty$ classifying p . We have the opposite coCartesian fibration $Rp : RM \rightarrow N(\Delta)^{\text{op}}$ that is classified by the functor $R\mathbf{X}$. We easily see that $Rp : RM \rightarrow N(\Delta)^{\text{op}}$ is a monoidal quasi-category. Note that the fiber $(RM)_{[n]}$ is equivalent to $(M_{[n]})^{\text{op}} \simeq (M_{[1]}^{\text{op}})^n$ for any $n \geq 0$. We say that RM is the opposite monoidal quasi-category of M .

A map $[m] \rightarrow [n]$ in Δ is said to be convex if it is injective and the image is $\{i, i+1, \dots, i+m\}$ for some i . Let $p : M \rightarrow N(\Delta)^{\text{op}}$ and $q : N \rightarrow N(\Delta)^{\text{op}}$ be monoidal

quasi-categories. A lax monoidal functor $G : N \rightarrow M$ between the monoidal quasi-categories is a map of simplicial sets over $N(\Delta)^{\text{op}}$ which carries p -coCartesian edges over convex morphisms in $N(\Delta)^{\text{op}}$ to q -coCartesian edges.

Lemma 1. *If $G_{[1]} : N_{[1]} \rightarrow M_{[1]}$ admits a left adjoint $F_{[1]}$, then there is a canonical functor $RF : RM \rightarrow RN$ over $N(\Delta)^{\text{op}}$ up to contractible space of choices. We have $(RF)_{[n]} \simeq (F_{[1]}^{\text{op}})^n$ for all $n \geq 0$.*

Proof. For any $n \geq 0$, we have equivalences $M_{[n]} \simeq (M_{[1]})^n$ and $N_{[n]} \simeq (N_{[1]})^n$. Since G is a lax monoidal functor, we see that $G_{[n]}$ is equivalent to $(G_{[1]})^n$ under the above equivalences. Hence $G_{[n]}$ admits a left adjoint for all $n \geq 0$. The lemma follows from Theorem 4. \square

Proposition 5. *If $G : N \rightarrow M$ is a lax monoidal functor between monoidal quasi-categories such that $G_{[1]} : N_{[1]} \rightarrow M_{[1]}$ admits a left adjoint, then the functor $RF : RM \rightarrow RN$ is also a lax monoidal functor between the opposite monoidal quasi-categories.*

Proof. We have to show that RF preserves coCartesian edges over convex morphisms. Let $\alpha : [m] \rightarrow [n]$ be a convex morphism in Δ . Since $G : N \rightarrow M$ is a lax monoidal functor, we have a commutative diagram

$$\begin{array}{ccc} N_{[n]} & \xrightarrow{G_{[n]}} & M_{[n]} \\ \alpha_!^N \downarrow & & \downarrow \alpha_!^M \\ N_{[m]} & \xrightarrow{G_{[m]}} & M_{[m]} \end{array}$$

in Cat_∞ . Since $\alpha_!^M : M_{[n]} \rightarrow M_{[m]}$ and $\alpha_!^N : N_{[n]} \rightarrow N_{[m]}$ are equivalent to projections $M_{[1]}^n \rightarrow M_{[1]}^m$ and $N_{[1]}^n \rightarrow N_{[1]}^m$, respectively, we see that the diagram is left adjointable. The proposition follows from Proposition 4. \square

3.3 Opposites of tensored quasi-categories over monoidal quasi-categories

In this subsection we study the opposite of a tensored quasi-category over a monoidal quasi-category. We show that the opposite of a lax tensored right adjoint functor between tensored quasi-categories induces a lax tensored right adjoint functor between the opposites of the tensored quasi-categories.

First, we recall the definition of left tensored quasi-category over a monoidal quasi-category. Let $p : X \rightarrow N(\Delta)^{\text{op}} \times \Delta^1$ be a coCartesian fibration of simplicial sets. For any $n \geq 0$, the identity $\text{id}_{[n]} : [n] \rightarrow [n]$ in Δ and the edge $\{0\} \rightarrow \{1\}$ in Δ^1 induces a morphism $([n], 0) \rightarrow ([n], 1)$ in $N(\Delta)^{\text{op}} \times \Delta^1$, and hence we obtain a functor of quasi-categories $\alpha_n : X_{([n], 0)} \rightarrow X_{([n], 1)}$. For any $n \geq 0$, the inclusion

$[0] \cong \{n\} \hookrightarrow [n]$ in Δ and the identity $\text{id}_{\{0\}} : \{0\} \rightarrow \{0\}$ in Δ^1 induces a morphism $([0], 0) \rightarrow ([n], 0)$ in $N(\Delta)^{\text{op}} \times \Delta^1$, and hence we obtain a functor of quasi-categories $\beta_n : X_{([n], 0)} \rightarrow X_{([0], 0)}$. If the base change of p along the inclusion $N(\Delta)^{\text{op}} \times \{1\} \hookrightarrow N(\Delta)^{\text{op}} \times \Delta^1$ is a monoidal quasi-category, and the functor

$$\alpha_n \times \beta_n : X_{([n], 0)} \longrightarrow X_{([n], 1)} \times X_{([0], 0)}$$

is a categorical equivalence for all $n \geq 0$, then we say that p is a left tensored quasi-category.

Now suppose $p : X \rightarrow N(\Delta)^{\text{op}} \times \Delta^1$ is a left tensored quasi-category. We set $\mathcal{M} = X_{([1], 1)}$ and $\mathcal{C} = X_{([0], 0)}$. Note that \mathcal{M} is the underlying quasi-category of a monoidal quasi-category $p|_{N(\Delta)^{\text{op}} \times \{1\}}$. We say that \mathcal{C} is left tensored over the monoidal quasi-category \mathcal{M} .

Let Σ be a set of edges of $N(\Delta)^{\text{op}} \times \Delta^1$ consisting of edges of the forms

$$([n], 0) \longrightarrow ([m], 0),$$

where $[m] \rightarrow [n]$ is a convex morphism in Δ that carries m to n , and

$$([n], i) \longrightarrow ([m], 1)$$

for $i = 0, 1$, where $[m] \rightarrow [n]$ is convex.

Suppose that $p : X \rightarrow N(\Delta)^{\text{op}} \times \Delta^1$ and $q : Y \rightarrow N(\Delta)^{\text{op}} \times \Delta^1$ are left tensored quasi-categories. We say that a functor $G : Y \rightarrow X$ over $N(\Delta)^{\text{op}} \times \Delta^1$ is a lax left tensored functor if G carries p -coCartesian edges over Σ to q -coCartesian edges.

Lemma 2. *Let $p : X \rightarrow N(\Delta)^{\text{op}} \times \Delta^1$ and $q : Y \rightarrow N(\Delta)^{\text{op}} \times \Delta^1$ be left tensored quasi-categories. If $G : Y \rightarrow X$ is a lax left tensored functor such that $G_{([0], 0)}$ and $G_{([1], 1)}$ admit left adjoints $F_{([0], 0)}$ and $F_{([1], 1)}$, respectively, then there is a canonical functor $RF : RX \rightarrow RY$ over $N(\Delta)^{\text{op}} \times \Delta^1$ up to contractible space of choices. We have $(RF)_{([n], 0)} \simeq (F_{([1], 1)}^{\text{op}})^n \times F_{([0], 0)}^{\text{op}}$ and $(RF)_{([n], 1)} \simeq (F_{([1], 1)}^{\text{op}})^n$ for all $n \geq 0$.*

Proof. For any $n \geq 0$, we have equivalences $X_{([n], 0)} \simeq (X_{([1], 1)})^n \times X_{([0], 0)}$ and $Y_{([n], 0)} \simeq (Y_{([1], 1)})^n \times Y_{([0], 0)}$. Since G is a lax left tensored functor, we see that $G_{([n], 0)}$ is equivalent to $(G_{([1], 1)})^n \times G_{([0], 0)}$. In the same way, we see that $G_{([n], 1)}$ is equivalent to $(G_{([1], 1)})^n$ for any $n \geq 0$. Hence G_s admits a left adjoint for all $s \in N(\Delta)^{\text{op}} \times \Delta^1$. The lemma follows from Theorem 4. \square

Proposition 6. *Let $p : X \rightarrow N(\Delta)^{\text{op}} \times \Delta^1$ and $q : Y \rightarrow N(\Delta)^{\text{op}} \times \Delta^1$ be left tensored quasi-categories. If $G : Y \rightarrow X$ is a lax left tensored functor such that $G_{([0], 0)}$ and $G_{([1], 1)}$ admit left adjoints, then the functor $RF : RX \rightarrow RY$ is also a lax left tensored functor.*

Proof. We can prove the proposition in the same way as Proposition 5. We have to show that RF preserves coCartesian edges over Σ . Let $\alpha : s \rightarrow s'$ be an edge in Σ . Since $G : Y \rightarrow X$ is a lax left tensored functor, we have a commutative diagram

$$\begin{array}{ccc}
Y_s & \xrightarrow{G_s} & X_s \\
\alpha_i^Y \downarrow & & \downarrow \alpha_i^X \\
Y_{s'} & \xrightarrow{G_{s'}} & X_{s'}
\end{array}$$

in Cat_∞ . Since $\alpha_i^Y : Y_s \rightarrow Y_{s'}$ and $\alpha_i^X : X_s \rightarrow X_{s'}$ are equivalent to the projections, we see that the diagram is left adjointable. The proposition follows from Proposition 4. \square

4 Quasi-categories of comodules

In this section we introduce a quasi-category of comodules over a coalgebra in a stable homotopy theory \mathcal{C} . We regard a coalgebra as an algebra object of the opposite monoidal quasi-category of A - A -bimodule objects, where A is an algebra object of \mathcal{C} . We regard a comodule object over a coalgebra Γ as a module object over Γ in the opposite quasi-category of A -module objects. We define a cotensor product of a right comodule and a left comodule over a coalgebra as a limit of the cobar construction. Using these formulations, we study the functor from the localization of \mathcal{C} with respect to A to the quasi-category of comodules over the coalgebra $A \otimes A$.

4.1 Monoidal structure on ${}_A\text{BMod}_A(\mathcal{C})^{\text{op}}$

In this subsection we introduce a quasi-category of coalgebras and a quasi-category of comodules over a coalgebra in a stable homotopy theory.

Let \mathcal{M}^\otimes be a monoidal quasi-category. We denote by \mathcal{M} the underlying quasi-category of the monoidal quasi-category \mathcal{M}^\otimes . For algebra objects A and B of \mathcal{M} , we denote by ${}_A\text{BMod}_B(\mathcal{M})$ the quasi-category of A - B -bimodule objects in \mathcal{M} . If B is the monoidal unit $\mathbf{1}$ in \mathcal{M} , we abbreviate the quasi-category ${}_A\text{BMod}_\mathbf{1}(\mathcal{M})$ of A - $\mathbf{1}$ -bimodule objects in \mathcal{M} as ${}_A\text{BMod}(\mathcal{M})$. Let \mathcal{N} be a quasi-category left tensored over \mathcal{M}^\otimes . For an algebra object A of \mathcal{M} , we denote by $\text{LMod}_A(\mathcal{N})$ the quasi-category of left A -module objects in \mathcal{N} . Note that there is a natural equivalence $\text{LMod}_A(\mathcal{M}) \simeq {}_A\text{BMod}(\mathcal{M})$ of quasi-categories.

Let $(\mathcal{C}, \otimes, \mathbf{1})$ be a stable homotopy theory in the sense of [23, Def. 2.1], that is, \mathcal{C} is a presentable stable quasi-category which is the underlying quasi-category of a symmetric monoidal quasi-category \mathcal{C}^\otimes , where the tensor product commutes with all colimits separately in each variable. For an algebra object A of \mathcal{C} , we denote by ${}_A\text{BMod}_A(\mathcal{C})$ the quasi-category of A - A -bimodules in \mathcal{C} , which is the underlying quasi-category of the monoidal quasi-category ${}_A\text{BMod}_A(\mathcal{C})^\otimes$, where the tensor product is given by the relative tensor product \otimes_A and the unit is the A - A -bimodule A (see [21, 4.3 and 4.4]). Note that the relative tensor product \otimes_A commutes with

all colimits separately in each variable by [21, Cor. 4.4.2.15]. For algebra objects A and B of \mathcal{C} , we denote by ${}_A\mathbf{BMod}_A(\mathcal{C})$ the quasi-category of A - B -bimodules, which is presentable by [21, Cor. 4.3.3.10].

If \mathcal{M}^\otimes is a monoidal quasi-category, then the opposite quasi-category $(\mathcal{M}^\otimes)^{\mathrm{op}}$ also carries a monoidal structure. Since ${}_A\mathbf{BMod}_A(\mathcal{C})$ is the underlying quasi-category of the monoidal quasi-category ${}_A\mathbf{BMod}_A(\mathcal{C})^\otimes$ for an algebra object A of \mathcal{C} , the opposite quasi-category ${}_A\mathbf{BMod}_A(\mathcal{C})^{\mathrm{op}}$ is the underlying quasi-category of the opposite monoidal quasi-category $({}_A\mathbf{BMod}_A(\mathcal{C})^\otimes)^{\mathrm{op}}$. We regard an algebra object Γ of ${}_A\mathbf{BMod}_A(\mathcal{C})^{\mathrm{op}}$ as a coalgebra object of ${}_A\mathbf{BMod}_A(\mathcal{C})$. We define the quasi-category ${}_A\mathbf{CoAlg}_A(\mathcal{C})$ of coalgebra objects of ${}_A\mathbf{BMod}_A(\mathcal{C})$ to be the opposite of the quasi-category of algebra objects of ${}_A\mathbf{BMod}_A(\mathcal{C})^{\mathrm{op}}$:

$${}_A\mathbf{CoAlg}_A(\mathcal{C}) = \mathbf{Alg}({}_A\mathbf{BMod}_A(\mathcal{C})^{\mathrm{op}})^{\mathrm{op}}.$$

For a quasi-category \mathcal{Y} left tensored over a monoidal category \mathcal{M}^\otimes , the opposite quasi-category $\mathcal{Y}^{\mathrm{op}}$ carries the structure of left tensored quasi-category over the opposite monoidal quasi-category $(\mathcal{M}^\otimes)^{\mathrm{op}}$.

The quasi-category ${}_A\mathbf{BMod}(\mathcal{C}) \simeq \mathbf{LMod}_A(\mathcal{C})$ is left tensored over ${}_A\mathbf{BMod}_A(\mathcal{C})^\otimes$ by the relative tensor product \otimes_A for an algebra object A of \mathcal{C} . Hence the opposite quasi-category ${}_A\mathbf{BMod}(\mathcal{C})^{\mathrm{op}}$ is left tensored over the opposite monoidal quasi-category $({}_A\mathbf{BMod}_A(\mathcal{C})^\otimes)^{\mathrm{op}}$. Let Γ be a coalgebra object of ${}_A\mathbf{BMod}_A(\mathcal{C})$, that is, an algebra object of ${}_A\mathbf{BMod}_A(\mathcal{C})^{\mathrm{op}}$. We regard a left Γ -module in ${}_A\mathbf{BMod}(\mathcal{C})^{\mathrm{op}}$ as a left Γ -comodule in ${}_A\mathbf{BMod}(\mathcal{C})$. We define the quasi-category of left Γ -comodules $\mathbf{LComod}_{(A,\Gamma)}(\mathcal{C})$ to be the opposite of the quasi-category of left Γ -module objects in ${}_A\mathbf{BMod}_A(\mathcal{C})^{\mathrm{op}}$:

$$\mathbf{LComod}_{(A,\Gamma)}(\mathcal{C}) = (\mathbf{LMod}_\Gamma({}_A\mathbf{BMod}(\mathcal{C})^{\mathrm{op}}))^{\mathrm{op}}.$$

Note that $\mathbf{LComod}_{(A,\Gamma)}(\mathcal{C})$ is right tensored over \mathcal{C} and there is a forgetful functor

$$\mathbf{LComod}_{(A,\Gamma)}(\mathcal{C}) \longrightarrow {}_A\mathbf{BMod}(\mathcal{C}) \simeq \mathbf{LMod}_A(\mathcal{C}),$$

which is a map of quasi-categories right tensored over \mathcal{C} .

In the same way as $\mathbf{LComod}_{(A,\Gamma)}(\mathcal{C})$, we can define the quasi-category of right Γ -comodules $\mathbf{RComod}_{(A,\Gamma)}(\mathcal{C})$ in \mathcal{C} for a coalgebra Γ of ${}_A\mathbf{BMod}_A(\mathcal{C})$.

4.2 Comparison maps

In this subsection we construct a functor between quasi-categories of comodules for a map of algebra objects. We also show that the definition of a quasi-category of comodules in this paper is consistent with the definition in [31].

Suppose $(\mathcal{C}, \otimes, \mathbf{1})$ be a stable homotopy theory. Let $f : A \rightarrow B$ be a map of algebra objects of \mathcal{C} . We have the functor

$$(f, f)^* : {}_B\mathbf{BMod}_B(\mathcal{C}) \longrightarrow {}_A\mathbf{BMod}_A(\mathcal{C})$$

which is obtained by restriction of scalars through f . The functor $(f, f)^*$ extends to a lax monoidal functor

$$((f, f)^*)^\otimes : {}_B\mathbf{BMod}_B(\mathcal{C})^\otimes \longrightarrow {}_A\mathbf{BMod}_A(\mathcal{C})^\otimes.$$

Furthermore, the functor $(f, f)^*$ admits a left adjoint

$$(f, f)_! : {}_A\mathbf{BMod}_A(\mathcal{C}) \longrightarrow {}_B\mathbf{BMod}_B(\mathcal{C}),$$

which assigns to an A - A -bimodule X the B - B -bimodule $B \otimes_A X \otimes_A B$. By Proposition 5, we obtain the following lemma.

Lemma 3. *If $f : A \rightarrow B$ is a map of algebra objects of \mathcal{C} , then the induced functor*

$$(f, f)_!^{\text{op}} : {}_A\mathbf{BMod}_A(\mathcal{C})^{\text{op}} \longrightarrow {}_B\mathbf{BMod}_B(\mathcal{C})^{\text{op}}$$

can be extended to a lax monoidal functor.

In the following of this paper, for simplicity, we say that the underlying quasi-category \mathcal{M} of a monoidal category \mathcal{M}^\otimes is a monoidal category and that the underlying functor $F : \mathcal{M} \rightarrow \mathcal{N}$ of a (lax) monoidal functor $F^\otimes : \mathcal{M}^\otimes \rightarrow \mathcal{N}^\otimes$ is a (lax) monoidal functor.

For a map of algebra objects $f : A \rightarrow B$ in \mathcal{C} , the lax monoidal functor $(f, f)_!^{\text{op}}$ induces a map of quasi-categories of algebra objects

$$(f, f)_!^{\text{op}} : \text{Alg}({}_A\mathbf{BMod}_A(\mathcal{C})^{\text{op}}) \longrightarrow \text{Alg}({}_B\mathbf{BMod}_B(\mathcal{C})^{\text{op}})$$

and hence we obtain a map of quasi-categories of coalgebra objects

$$(f, f)_! : {}_A\mathbf{CoAlg}_A(\mathcal{C}) \longrightarrow {}_B\mathbf{CoAlg}_B(\mathcal{C}).$$

Therefore, for a coalgebra object Γ of ${}_A\mathbf{BMod}_A(\mathcal{C})$, we obtain a coalgebra object

$$B \otimes_A \Gamma \otimes_A B = (f, f)_!(\Gamma)$$

of ${}_B\mathbf{BMod}_B(\mathcal{C})$. In particular, since the monoidal unit A in ${}_A\mathbf{BMod}_A(\mathcal{C})$ is a coalgebra object, we see that

$$B \otimes_A B = (f, f)_!(A)$$

is a coalgebra object of ${}_B\mathbf{BMod}_B(\mathcal{C})$.

In particular, since the monoidal unit $\mathbf{1}$ is a coalgebra object of ${}_1\mathbf{BMod}_1(\mathcal{C}) \simeq \mathcal{C}$, we have a coalgebra object

$$A \otimes A = (f, f)_!(\mathbf{1})$$

of ${}_A\mathbf{BMod}_A(\mathcal{C})$, where $f : \mathbf{1} \rightarrow A$ is the unit map of A . We write

$$\Gamma(A) = (A, A \otimes A)$$

for simplicity and we call $A \otimes A$ -comodules $\Gamma(A)$ -comodules interchangeably.

Let $f : A \rightarrow B$ be a map of algebra objects of \mathcal{C} . We denote by $f^* : \mathbf{LMod}_B(\mathcal{C}) \rightarrow \mathbf{LMod}_A(\mathcal{C})$ the restriction of scalars functor. Recall that f^* is a right adjoint to the extension of scalars functor

$$f_! : \mathbf{LMod}_A(\mathcal{C}) \rightarrow \mathbf{LMod}_B(\mathcal{C}),$$

which is given by $f_!(M) \simeq B \otimes_A M$.

Theorem 5. *Let Γ be a coalgebra object in ${}_A\mathbf{BMod}_A(\mathcal{C})$ and let $f : A \rightarrow B$ be a map of algebra objects of \mathcal{C} . The map f induces a functor of quasi-categories*

$$f_! : \mathbf{LComod}_{(A,\Gamma)}(\mathcal{C}) \longrightarrow \mathbf{LComod}_{(B,\Sigma)}(\mathcal{C})$$

which covers the functor $f_! : \mathbf{LMod}_A(\mathcal{C}) \rightarrow \mathbf{LMod}_B(\mathcal{C})$ through the forgetful functors, where $\Sigma = (f, f)_! \Gamma$.

Proof. This follows from Proposition 6. \square

Suppose we have a map $f : A \rightarrow B$ of algebra objects of \mathcal{C} . This induces an adjunction of functors

$$f_! : \mathbf{LMod}_A(\mathcal{C}) \rightleftarrows \mathbf{LMod}_B(\mathcal{C}) : f^*.$$

Taking the opposite quasi-categories, we obtain an adjunction of functors

$$f^{*\text{op}} : \mathbf{LMod}_B(\mathcal{C})^{\text{op}} \rightleftarrows \mathbf{LMod}_A(\mathcal{C})^{\text{op}} : f_!^{\text{op}}.$$

By this adjunction, we obtain an endomorphism monad

$$T \in \mathbf{Alg}(\mathbf{End}(\mathbf{LMod}_B(\mathcal{C})^{\text{op}})),$$

and a quasi-category

$$\mathbf{LMod}_T(\mathbf{LMod}_B(\mathcal{C})^{\text{op}}).$$

of left T -modules in $\mathbf{LMod}_B(\mathcal{C})^{\text{op}}$ (see [21, §4.7.4]).

The following theorem shows that the definition of a quasi-category of comodules is consistent with the definition in [31].

Theorem 6. *There is an equivalence of quasi-categories*

$$\mathbf{LComod}_{(B, B \otimes_A B)}(\mathcal{C}) \simeq \mathbf{LMod}_T(\mathbf{LMod}_B(\mathcal{C})^{\text{op}})^{\text{op}}.$$

Proof. Put $\mathcal{A} = \mathbf{LMod}_A(\mathcal{C})^{\text{op}}$ and $\mathcal{B} = \mathbf{LMod}_B(\mathcal{C})^{\text{op}}$. We have an adjunction of functors $F : \mathcal{B} \rightleftarrows \mathcal{A} : G$, where $F = f^{*\text{op}}$ and $G = f_!^{\text{op}}$. Since $\mathbf{LMod}_B(\mathcal{C}) \simeq {}_B\mathbf{BMod}(\mathcal{C})$, we can regard $B \otimes_A B$ as an algebra object of $\mathbf{End}(\mathcal{B})$. By [21, Prop. 4.7.4.3], we can lift G to $\bar{G} \in \mathbf{LMod}_{B \otimes_A B}(\mathbf{Fun}(\mathcal{B}, \mathcal{A}))$. We can verify that the composition

$$B \otimes_A B \longrightarrow (B \otimes_A B) \circ G \circ F \longrightarrow G \circ F$$

is an equivalence in $\text{End}(\mathcal{B})$, where the first map is induced by the unit map of the adjunction (F, G) , and the second map is induced by the left $B \otimes_A B$ -action on G in $\text{Fun}(\mathcal{B}, \mathcal{A})$. By [21, Prop. 4.7.4.3], we see that $B \otimes_A B$ is equivalent to the endomorphism monad T . Hence we obtain an equivalence between $\text{LMod}_T(\text{LMod}_B(\mathcal{C})^{\text{op}})$ and $\text{LMod}_{B \otimes_A B}({}_B \text{BMod}(\mathcal{C})^{\text{op}})$. \square

4.3 Cotensor products for comodules in quasi-categories

Let $(\mathcal{C}, \otimes, \mathbf{1})$ be a stable homotopy theory. In this subsection we define a (derived) cotensor product of comodules in \mathcal{C} . In particular, we define a (derived) functor of taking primitives of comodules. We also study a comodule structure on cotensor products.

Let A be an algebra object of \mathcal{C} . Suppose Γ is a coalgebra object of the quasi-category ${}_A \text{BMod}_A(\mathcal{C})$ of A - A -bimodules in \mathcal{C} , that is, Γ is an algebra object of the opposite monoidal quasi-category ${}_A \text{BMod}_A(\mathcal{C})^{\text{op}}$.

For a right Γ -comodule M and a left Γ -comodule N , we shall define a cotensor product

$$M \square_{\Gamma} N.$$

We regard M as an object in $\text{RMod}_{\Gamma}(\text{BMod}_A(\mathcal{C})^{\text{op}})$ and N as an object in $\text{LMod}_{\Gamma}({}_A \text{BMod}(\mathcal{C})^{\text{op}})$. We can construct a two-sided bar construction

$$B_{\bullet}(M, \Gamma, N),$$

which is a simplicial object in \mathcal{C}^{op} . The simplicial object $B_{\bullet}(M, \Gamma, N)$ has the n th term given by

$$B_n(M, \Gamma, N) \simeq M \otimes_A \overbrace{\Gamma \otimes_A \cdots \otimes_A \Gamma}^n \otimes_A N$$

with the usual structure maps. We regard $B_{\bullet}(M, \Gamma, N)$ as a cosimplicial object

$$C^{\bullet}(M, \Gamma, N)$$

in \mathcal{C} and define the cotensor product $M \square_{\Gamma} N$ to be the limit of the cosimplicial object $C^{\bullet}(M, \Gamma, N)$:

$$M \square_{\Gamma} N = \lim_{N(\Delta)} C^{\bullet}(M, \Gamma, N).$$

Now we regard A as a right A -module and suppose that A is a right Γ -comodule via $\eta_R : A \rightarrow A \otimes_A \Gamma \simeq \Gamma$. We define a functor

$$P : \text{LComod}_{(A, \Gamma)} \longrightarrow \mathcal{C}$$

by

$$P(N) = A \square_{\Gamma} N.$$

We consider the functor P is a derived functor of taking primitives in N .

Suppose we have algebra objects A, B, C of \mathcal{C} . The quasi-category of B - A -bimodules ${}_B\mathbf{BMod}_A(\mathcal{C})$ in \mathcal{C} is right tensored over the monoidal quasi-category ${}_A\mathbf{BMod}_A(\mathcal{C})$ and the quasi-category of A - C -bimodules ${}_A\mathbf{BMod}_C(\mathcal{C})$ in \mathcal{C} is left tensored over the monoidal quasi-category ${}_A\mathbf{BMod}_A(\mathcal{C})$. Let Γ be a coalgebra object of ${}_A\mathbf{BMod}_A(\mathcal{C})$. We can define right Γ -comodule objects of ${}_B\mathbf{BMod}_A(\mathcal{C})$ and left Γ -comodule objects of ${}_A\mathbf{BMod}_C(\mathcal{C})$ in the same way as Γ -comodule objects of ${}_A\mathbf{BMod}_A(\mathcal{C})$. Suppose we have a right Γ -comodule M of ${}_B\mathbf{BMod}_A(\mathcal{C})$ and a left Γ -comodule N of ${}_A\mathbf{BMod}_C(\mathcal{C})$. We can form the cobar construction $C^\bullet(M, \Gamma, N)$ in ${}_B\mathbf{BMod}_C(\mathcal{C})$. Hence the cotensor product $M \square_\Gamma N$ is a B - C -bimodule

$$M \square_\Gamma N \in {}_B\mathbf{BMod}_C(\mathcal{C}).$$

Let Σ be a coalgebra object of ${}_B\mathbf{BMod}_B(\mathcal{C})$. Now suppose M is a (Σ, Γ) -bicomodule object of ${}_B\mathbf{BMod}_A(\mathcal{C})$, that is, M is a (Σ, Γ) -bimodule object of ${}_B\mathbf{BMod}_A(\mathcal{C})^{\text{op}}$. In general, the cotensor product $M \square_\Gamma N$ does not support a left Σ -comodule structure. The following proposition gives us a sufficient condition for $M \square_\Gamma N$ to be a left Σ -comodule object of ${}_B\mathbf{BMod}_C(\mathcal{C})$ induced by the left Σ -comodule structure on M .

Proposition 7. *Let M be a (Σ, Γ) -bicomodule object of ${}_B\mathbf{BMod}_A(\mathcal{C})$ and let N be a left Γ -comodule object of ${}_A\mathbf{BMod}_C(\mathcal{C})$. If the canonical map*

$$\overbrace{\Sigma \otimes_B \cdots \otimes_B \Sigma}^r \otimes_B (M \square_\Gamma N) \longrightarrow (\overbrace{\Sigma \otimes_B \cdots \otimes_B \Sigma}^r \otimes_B M) \square_\Gamma N$$

is an equivalence in ${}_B\mathbf{BMod}_C(\mathcal{C})$ for all $r > 0$, then the left Σ -comodule structure on M induces a left Σ -comodule structure on $M \square_\Gamma N$.

In order to prove Proposition 7, we need the following lemma.

Lemma 4. *Let \mathcal{M} be a monoidal quasi-category, A an algebra object of \mathcal{M} , and \mathcal{D} a quasi-category left-tensored over \mathcal{M} . Suppose we have a diagram $X : K \rightarrow \mathbf{LMod}_A(\mathcal{D})$, where K is a simplicial set. We set $Y = \pi \circ X : K \rightarrow \mathcal{D}$, where $\pi : \mathbf{LMod}_A(\mathcal{D}) \rightarrow \mathcal{D}$ is the forgetful functor. We assume that there exists a colimit $\text{colim}_K^{\mathcal{D}}(A^{\otimes r} \otimes Y)$ in \mathcal{D} for all $r \geq 0$. If the canonical map $\text{colim}_K^{\mathcal{D}}(A^{\otimes r} \otimes Y) \rightarrow A^{\otimes r} \otimes \text{colim}_K^{\mathcal{D}} Y$ is an equivalence for all $r > 0$, then there exists a colimit of X in $\mathbf{LMod}_A(\mathcal{D})$ and the forgetful functor $\pi : \mathbf{LMod}_A(\mathcal{D}) \rightarrow \mathcal{D}$ preserves the colimit.*

Proof. We use the notation in [21, §4.2.2]. Let \mathcal{D}^{\otimes} and \mathcal{M}^{\otimes} be quasi-categories defined in [21, Notation 4.2.2.16]. We have maps $\mathcal{D}^{\otimes} \xrightarrow{q} \mathcal{M}^{\otimes} \xrightarrow{p} N(\Delta)^{\text{op}}$, where p and $p \circ q$ are coCartesian fibrations by [21, Remark. 4.2.2.24]. Furthermore, q is a categorical fibration by [21, Remark. 4.2.2.18] and a locally coCartesian fibration by [21, Lem. 4.2.2.19]. Note that there is an equivalence of quasi-categories $\mathcal{D}_{[s]}^{\otimes} \simeq \mathcal{M}_{[s]}^{\otimes} \times \mathcal{D}$ and the restriction $q_{[s]} : \mathcal{D}_{[s]}^{\otimes} \rightarrow \mathcal{M}_{[s]}^{\otimes}$ is the projection for any $[s] \in N(\Delta)^{\text{op}}$.

We have simplicial models of algebra and module objects in quasi-categories (see [21, §4.1.2 and §4.2.2]). We have a full subcategory ${}^A\mathbf{Alg}(\mathcal{M})$ of the quasi-category

$\text{Fun}_{N(\Delta)^{\text{op}}}(N(\Delta)^{\text{op}}, \mathcal{M}^{\otimes})$ which is equivalent to the quasi-category $\text{Alg}(\mathcal{M})$ of algebra objects of \mathcal{M} (see [21, Def. 4.1.2.14 and Prop. 4.1.2.15]). We denote by $A' : N(\Delta)^{\text{op}} \rightarrow \mathcal{M}^{\otimes}$ the corresponding simplicial object of \mathcal{M}^{\otimes} to $A \in \text{Alg}(\mathcal{M})$.

We form a pullback diagram

$$\begin{array}{ccc} \mathcal{N} & \xrightarrow{j} & \mathcal{D}^{\otimes} \\ q' \downarrow & & \downarrow q \\ N(\Delta)^{\text{op}} & \xrightarrow{A'} & \mathcal{M}^{\otimes}, \end{array}$$

where q' is a locally coCartesian fibration and a categorical fibration. Note that the fiber $\mathcal{N}_{[n]}$ of q' over $[n]$ is equivalent to \mathcal{D} for all $[n] \in N(\Delta)^{\text{op}}$. We have a full subcategory ${}^{\Delta}\text{LMod}_{A'}(\mathcal{D})$ of $\text{Fun}_{N(\Delta)^{\text{op}}}(N(\Delta)^{\text{op}}, \mathcal{N})$, which is equivalent to $\text{LMod}_A(\mathcal{D})$ (see [21, Cor. 4.2.2.15]). An object G of $\text{Fun}_{N(\Delta)^{\text{op}}}(N(\Delta)^{\text{op}}, \mathcal{N})$ belongs to $\text{LMod}_{A'}(\mathcal{D})$ if and only if the edge $(j \circ G)(\alpha^{\text{op}}) : (j \circ G)([n]) \rightarrow (j \circ G)([m])$ is $p \circ q$ -coCartesian for any convex map $\alpha : [m] \rightarrow [n]$ in Δ such that $\alpha(m) = n$.

We denote by $f : K \times N(\Delta)^{\text{op}} \rightarrow \mathcal{N}$ the map corresponding to the diagram $X \in \text{Fun}(K, \text{LMod}_A(\mathcal{D}))$. We let $g : K^{\triangleright} \times N(\Delta)^{\text{op}} \rightarrow N(\Delta)^{\text{op}}$ be the projection. We have a commutative diagram

$$\begin{array}{ccc} K \times N(\Delta)^{\text{op}} & \xrightarrow{f} & \mathcal{N} \\ \downarrow & & \downarrow q' \\ K^{\triangleright} \times N(\Delta)^{\text{op}} & \xrightarrow{g} & N(\Delta)^{\text{op}}, \end{array}$$

where the left vertical arrow is the inclusion. We shall show that there is a q' -left Kan extension $\bar{f} : K^{\triangleright} \times N(\Delta)^{\text{op}} \rightarrow \mathcal{N}$ which makes the whole diagram commutative, and that the adjoint map gives rise to a colimit diagram $K^{\triangleright} \rightarrow \text{LMod}_{A'}(\mathcal{D}) \simeq \text{LMod}_A(\mathcal{D})$.

Let $f_{[n]} : K \rightarrow \mathcal{N}_{[n]} \simeq \mathcal{D}$ be the restriction of the map f over $[n] \in N(\Delta)^{\text{op}}$, which is equivalent to Y . Since Y has a colimit in \mathcal{D} by the assumption, we obtain a colimit diagram $\bar{f}_{[n]} : K^{\triangleright} \rightarrow \mathcal{N}_{[n]} \simeq \mathcal{D}$ that is an extension of f .

Let $\alpha : [n] \rightarrow [m]$ be an edge in $N(\Delta)^{\text{op}}$. Since q' is a locally coCartesian fibration, α induces a functor $\alpha_! : \mathcal{N}_{[n]} \rightarrow \mathcal{N}_{[m]}$. The composition $\alpha_! \circ f_{[n]} : K \rightarrow \mathcal{N}_{[m]} \simeq \mathcal{D}$ is equivalent to $A^{\otimes r} \otimes Y$ for some $r \geq 0$. This implies that $\alpha_! \circ \bar{f}_{[n]}$ is a colimit diagram in $\mathcal{N}_{[m]} \simeq \mathcal{D}$ by the assumption that the canonical map $\text{colim}_K^{\mathcal{D}}(A^{\otimes r} \otimes Y) \rightarrow A^{\otimes r} \otimes \text{colim}_K^{\mathcal{D}} Y$ is an equivalence. Hence we see that $i_{[n]} \circ \bar{f}_{[n]}$ is a q' -colimit diagram by [20, Prop. 4.3.1.10], where $i_{[n]} : \mathcal{N}_{[n]} \hookrightarrow \mathcal{N}$ is the inclusion.

By the dual of [21, Lem. 3.2.2.9(1)], there exists a q' -left Kan extension $\bar{f} : K^{\triangleright} \times N(\Delta)^{\text{op}} \rightarrow \mathcal{N}$ of f such that $q' \circ \bar{f} = g$. The restriction of \bar{f} to $K^{\triangleright} \times \{[n]\}$ is equivalent to $i_{[n]} \circ \bar{f}_{[n]}$ for all $[n] \in N(\Delta)^{\text{op}}$.

We consider the adjoint map $K^{\triangleright} \rightarrow \text{Fun}(N(\Delta)^{\text{op}}, \mathcal{N})$ of f . By the dual of [21, Lem. 3.2.2.9(2)], this map is a $(q')^{N(\Delta)^{\text{op}}}$ -colimit diagram, where $(q')^{N(\Delta)^{\text{op}}} :$

$\text{Fun}(N(\Delta)^{\text{op}}, \mathcal{N}) \rightarrow \text{Fun}(N(\Delta)^{\text{op}}, N(\Delta)^{\text{op}})$ is induced by q' . Since g is the projection, we see that it factors through $\text{Fun}_{N(\Delta)^{\text{op}}}(N(\Delta)^{\text{op}}, \mathcal{N})$ and we obtain a map $\widehat{f} : K^\triangleright \rightarrow \text{Fun}_{N(\Delta)^{\text{op}}}(N(\Delta)^{\text{op}}, \mathcal{N})$. By [20, Prop. 4.3.1.5(4)], we see that \widehat{f} is a colimit diagram.

We shall show that \widehat{f} factors through $\text{LMod}_{A'}(\mathcal{D})$. Note that the restriction of \widehat{f} to K factors through $\text{LMod}_{A'}(\mathcal{D})$. Let $F = \widehat{f}(\infty) \in \text{Fun}_{N(\Delta)^{\text{op}}}(N(\Delta)^{\text{op}}, \mathcal{N})$, where ∞ is the cone point of K^\triangleright . Since $\widehat{f}_{[n]}$ is a colimit diagram extending Y , we have $F([n]) \simeq \text{colim}_K^{\mathcal{D}} Y$ in $\mathcal{N}_{[n]} \simeq \mathcal{D}$ for any $[n] \in N(\Delta)^{\text{op}}$. Let $\alpha : [m] \rightarrow [n]$ be a convex map in Δ such that $\alpha(m) = n$. The induced functor $\alpha_! : \mathcal{N}_{[n]} \rightarrow \mathcal{N}_{[m]}$ is identified with the identity functor of \mathcal{D} . This implies that $(j \circ F)(\alpha^{\text{op}}) : (j \circ F)([n]) \rightarrow (j \circ F)([m])$ is a $p \circ q$ -coCartesian edge. Hence \widehat{f} factors through the full subcategory ${}^A\text{LMod}_{A'}(\mathcal{D})$ and the map $\widehat{f} : K^\triangleright \rightarrow {}^A\text{LMod}_{A'}(\mathcal{D})$ is a colimit diagram.

By the construction of \widehat{f} , the composition $\pi \circ \widehat{f} : K^\triangleright \rightarrow \mathcal{D}$ is also a colimit diagram, where $\pi : {}^A\text{LMod}_{A'}(\mathcal{D}) \rightarrow \mathcal{D}$ is the forgetful functor. This completes the proof. \square

Proof (Proof of Proposition 7). We shall apply Lemma 4. We have the monoidal quasi-category ${}_B\text{BMod}_B(\mathcal{C})^{\text{op}}$, the algebra object Σ of ${}_B\text{BMod}_B(\mathcal{C})^{\text{op}}$, and the quasi-category ${}_B\text{BMod}_C(\mathcal{C})^{\text{op}}$ left tensored over ${}_B\text{BMod}_B(\mathcal{C})^{\text{op}}$. By the bar construction, we have a simplicial object $B_\bullet(M, \Gamma, N)$ of $\text{LMod}_\Sigma({}_B\text{BMod}_C(\mathcal{C})^{\text{op}})$. By the assumption, the canonical map

$$\text{colim}_{N(\Delta)^{\text{op}}} B_\bullet(\Sigma^{\otimes Br} \otimes_B M, \Gamma, N) \rightarrow \Sigma^{\otimes Br} \otimes_B \text{colim}_{N(\Delta)^{\text{op}}} B_\bullet(M, \Gamma, N)$$

is an equivalence in ${}_B\text{BMod}_C(\mathcal{C})^{\text{op}}$ for all $r > 0$. By Lemma 4, there exists a colimit of $B_\bullet(M, \Gamma, N)$ in $\text{LMod}_\Sigma({}_B\text{BMod}_C(\mathcal{C})^{\text{op}})$ and the colimit is created in ${}_B\text{BMod}_C(\mathcal{C})^{\text{op}}$. Hence we see that the cosimplicial object $C^\bullet(M, \Gamma, N)$ has a limit in $\text{LComod}_\Sigma({}_B\text{BMod}_C(\mathcal{C}))$ and the underlying object of the limit is $M \square_\Gamma N$ in ${}_B\text{BMod}_C(\mathcal{C})$. \square

4.4 Equivalence of quasi-categories of comodules

Let $(\mathcal{C}, \otimes, \mathbf{1})$ be a stable homotopy theory and let A be an algebra object of \mathcal{C} . In this subsection we study the relationship between the localization of \mathcal{C} with respect to A and the quasi-category of $\Gamma(A)$ -comodules in \mathcal{C} .

We regard A as a right A -module and the map of right A -modules $\eta_R : A \simeq S \otimes A \rightarrow A \otimes A$ induces a right $\Gamma(A)$ -comodule structure on A . Since $\text{RComod}_{\Gamma(A)}(\mathcal{C})$ is left tensored over \mathcal{C} , we have $X \otimes A \in \text{RComod}_{\Gamma(A)}(\mathcal{C})$ for any $X \in \mathcal{C}$.

Recall that we have a cosimplicial object

$$C^\bullet(A, A \otimes A, M)$$

in \mathcal{C} by cobar construction for $M \in \mathrm{LComod}_{\Gamma(A)}(\mathcal{C})$. The totalization of $C^\bullet(A, A \otimes A, M)$ is $P(M)$. In the same way as in [31, Prop. 5.1], we obtain an adjunction of functors

$$A \otimes (-) : \mathcal{C} \rightleftarrows \mathrm{LComod}_{\Gamma(A)}(\mathcal{C}) : P.$$

Let $C^\bullet : N(\Delta) \rightarrow \mathcal{C}$ be a cosimplicial object in \mathcal{C} . We recall the Tot tower associated to C^\bullet . For $r \geq 0$, we denote by $\Delta^{\leq r}$ the full subcategory of Δ spanned by $\{[0], [1], \dots, [r]\}$. We denote by $C^\bullet|_{N(\Delta^{\leq r})}$ the restriction of C^\bullet to $N(\Delta^{\leq r})$. We recall that $\mathrm{Tot}_r(C^\bullet)$ is defined to be the limit of $C^\bullet|_{N(\Delta^{\leq r})}$ in \mathcal{C} . The inclusion $\Delta^{\leq r} \hookrightarrow \Delta^{\leq r+1}$ induces a map $\mathrm{Tot}_{r+1}(C^\bullet) \rightarrow \mathrm{Tot}_r(C^\bullet)$ for $r \geq 0$ and we obtain a tower $\{\mathrm{Tot}_r(C^\bullet)\}_{r \geq 0}$. Note that the limit of the tower is equivalent to $\mathrm{Tot}(C^\bullet)$:

$$\mathrm{Tot}(C^\bullet) \simeq \lim_r \mathrm{Tot}_r(C^\bullet).$$

If there is a coaugmentation $D \rightarrow C^\bullet$, then we obtain a map of towers $c(D) \rightarrow \{\mathrm{Tot}_r(C^\bullet)\}_{r \geq 0}$, where $c(D)$ is the constant tower on D .

We denote by $\mathrm{Pro}(\mathcal{C})$ the quasi-category of pro-objects in \mathcal{C} (see [23, §3]). We have a fully faithful embedding $\mathcal{C} \hookrightarrow \mathrm{Pro}(\mathcal{C})$. We say that an object of $\mathrm{Pro}(\mathcal{C})$ is constant if it is equivalent to an object in the image of the embedding $\mathcal{C} \hookrightarrow \mathrm{Pro}(\mathcal{C})$.

Lemma 5. *For any $M \in \mathrm{LComod}_{\Gamma(A)}(\mathcal{C})$, the cosimplicial object $A \otimes C^\bullet(A, A \otimes A, M)$ is split, and hence the tower $\{\mathrm{Tot}^r(A \otimes C^\bullet(A, A \otimes A, M))\}$ associated to the cosimplicial object $A \otimes C^\bullet(A, A \otimes A, M)$ is equivalent to the constant object M in $\mathrm{Pro}(\mathcal{C})$.*

Proof. We have an isomorphism of cosimplicial objects

$$A \otimes C^\bullet(A, A \otimes A, M) \cong C^\bullet(A \otimes A, A \otimes A, M).$$

The lemma follows from the fact that $C^\bullet(A \otimes A, A \otimes A, M)$ is a split cosimplicial object. \square

A full subcategory $\mathcal{I} \subset \mathcal{C}$ is said to be an ideal if $X \otimes Y \in \mathcal{I}$ whenever $X \in \mathcal{I}$ and $Y \in \mathcal{C}$ (cf. [23, Definition 2.16]). A full subcategory $\mathcal{D} \subset \mathcal{C}$ is said to be thick if \mathcal{D} is closed under finite limits and colimits and under retracts. If, furthermore, \mathcal{D} is an ideal, we say that \mathcal{D} is a thick tensor ideal (cf. [23, Definition 3.16]). Given a collection of objects in \mathcal{C} , the thick tensor ideal generated by them is the smallest thick tensor ideal containing the collection. Let A be an algebra object of \mathcal{C} . An object $X \in \mathcal{C}$ is said to be A -nilpotent if X belongs to the thick tensor ideal generated by A .

Lemma 6. *Let A be an algebra object of \mathcal{C} . If the unit $\mathbf{1}$ is A -nilpotent, then the tower associated to the cosimplicial object $C^\bullet(A, A \otimes A, M)$ is equivalent to the constant object $P(M)$ in $\mathrm{Pro}(\mathcal{C})$ for any $M \in \mathrm{LComod}_{\Gamma(A)}(\mathcal{C})$.*

Proof. Let \mathcal{I} be the class of objects X in \mathcal{C} such that the tower associated to the cosimplicial object $X \otimes C^\bullet(A, A \otimes A, M)$ is equivalent to a constant object in $\mathrm{Pro}(\mathcal{C})$.

We see that \mathcal{J} is a thick tensor ideal of \mathcal{C} and contains A by Lemma 5. Hence \mathcal{J} contains the unit $\mathbf{1}$ by the assumption. This implies that the tower associated to $C^\bullet(A, A \otimes A, M)$ is equivalent to the constant object $\lim_{N(\Delta)} C^\bullet(A, A \otimes A, M) \simeq P(M)$ in $\text{Pro}(\mathcal{C})$. \square

For any $X \in \mathcal{C}$ and $M \in \text{LComod}_{\Gamma(A)}(\mathcal{C})$, we have an equivalence of cosimplicial objects $X \otimes C^\bullet(A, A \otimes A, M) \simeq C^\bullet(X \otimes A, A \otimes A, M)$. This induces a natural map

$$X \otimes P(M) \longrightarrow (X \otimes A) \square_{A \otimes A} M.$$

Proposition 8. *Let A be an algebra object of \mathcal{C} . If the unit $\mathbf{1}$ is A -nilpotent, then the natural map $X \otimes P(M) \rightarrow (X \otimes A) \square_{A \otimes A} M$ is an equivalence for any $X \in \mathcal{C}$ and $M \in \text{Comod}_{\Gamma(A)}(\mathcal{C})$.*

Proof. By Lemma 6, the tower associated to the cosimplicial object $C^\bullet(A, A \otimes A, M)$ is equivalent to the constant object $P(M)$ in $\text{Pro}(\mathcal{C})$. This implies that the tower

$$\{\text{Tot}^r(X \otimes C^\bullet(A, A \otimes A, M))\}$$

associated to the cosimplicial object $X \otimes C^\bullet(A, A \otimes A, M)$ is also equivalent to the constant object $X \otimes P(M)$ in $\text{Pro}(\mathcal{C})$. By the equivalence of cosimplicial objects $X \otimes C^\bullet(A, A \otimes A, M) \simeq C^\bullet(X \otimes A, A \otimes A, M)$, we obtain the equivalence $X \otimes P(M) \simeq (X \otimes A) \square_{A \otimes A} M$. \square

Corollary 1. *Let A be an algebra object of \mathcal{C} . If the unit $\mathbf{1}$ is A -nilpotent, then the counit map*

$$A \otimes P(M) \rightarrow M$$

is an equivalence for any $M \in \text{LComod}_{\Gamma(A)}(\mathcal{C})$.

Proof. By Proposition 8, we have a natural equivalence $A \otimes P(M) \simeq (A \otimes A) \square_{A \otimes A} M$. The corollary follows from the fact that $(A \otimes A) \square_{A \otimes A} M \simeq M$. \square

We consider the localization of \mathcal{C} with respect to A . A morphism $f : X \rightarrow Y$ in \mathcal{C} is said to be an A -equivalence if $A \otimes f$ is an equivalence. We denote by $L_A \mathcal{C}$ the localization of \mathcal{C} with respect to the class of A -equivalences.

The following theorem is a slight generalization of [23, Prop. 3.21].

Theorem 7. *Let A be an algebra object of \mathcal{C} . If the unit $\mathbf{1}$ is A -nilpotent, then $L_A \mathcal{C}$ is equivalent to $\text{LComod}_{\Gamma(A)}(\mathcal{C})$. We have an adjoint equivalence*

$$A \otimes (-) : L_A \mathcal{C} \rightleftarrows \text{LComod}_{\Gamma(A)}(\mathcal{C}) : P,$$

and we can identify the functor $A \otimes (-) : \mathcal{C} \rightarrow \text{LComod}_{\Gamma(A)}(\mathcal{C})$ with the localization $\mathcal{C} \rightarrow L_A \mathcal{C}$.

Proof. We have an adjoint pair of functors $A \otimes (-) : \mathcal{C} \rightleftarrows \text{LComod}_{\Gamma(A)}(\mathcal{C}) : P$. Clearly, $A \otimes f$ is an equivalence in $\text{LComod}_{\Gamma(A)}(\mathcal{C})$ if and only if f is an A -equivalence for any morphism f in \mathcal{C} . Hence it suffices to show that the right adjoint P is fully faithful. The counit map $\varepsilon : A \otimes P(M) \rightarrow M$ is an equivalence for any $M \in \text{LComod}_{\Gamma(A)}(\mathcal{C})$ by Corollary 1. Hence we see that P is fully faithful. \square

5 Comodules in the quasi-category of spectra

In this section we study quasi-category of comodules in spectra. Using the Bousfield-Kan spectral sequence and the results in [17], we show that the quasi-category of comodules associated to a Landweber exact \mathbb{S} -algebra depends only on its height. We also show that the $E(n)$ -local category is equivalent to the quasi-category of comodules over the coalgebra $E(n) \otimes E(n)$. In [31] we considered the model category of F_n -modules in the category of discrete symmetric \mathbb{G}_n -spectra, where \mathbb{G}_n is the extended Morava stabilizer group and F_n is a discrete model of the Morava E -theory spectrum E_n . We show that the category of F_n -modules in the discrete symmetric \mathbb{G}_n -spectra models the $K(n)$ -local category.

5.1 Cotensor product and its derived functor in algebraic setting

In this subsection we recall some properties of the category of comodules over a coalgebra in algebraic setting. We study the derived functor of cotensor product of comodules and show that the derived functor can be described by the cobar complex in some situations. The content in this section is not new. Our main reference is [29, Appendix A1.2].

Let \mathbf{Ab}_* be the category of graded abelian groups. Let A be a monoid object in \mathbf{Ab}_* . We denote by ${}_A\mathbf{BMod}_A(\mathbf{Ab}_*)$ the category of A - A -bimodules in \mathbf{Ab}_* . The category ${}_A\mathbf{BMod}_A(\mathbf{Ab}_*)$ is a monoidal category with the tensor product \otimes_A and the unit A . We denote by ${}_A\mathbf{BMod}_A(\mathbf{Ab}_*)^{\text{op}}$ the opposite monoidal category. A coalgebra in ${}_A\mathbf{BMod}_A(\mathbf{Ab}_*)$ is defined to be a monoid object in ${}_A\mathbf{BMod}_A(\mathbf{Ab}_*)^{\text{op}}$. In other word, a coalgebra Γ is an A - A -bimodule equipped with maps

$$\begin{aligned}\psi : \Gamma &\longrightarrow \Gamma \otimes_A \Gamma, \\ \varepsilon : \Gamma &\longrightarrow A\end{aligned}$$

in ${}_A\mathbf{BMod}_A(\mathbf{Ab}_*)$ satisfying the coassociativity and counit conditions.

We denote by ${}_A\mathbf{CoAlg}_A(\mathbf{Ab}_*)$ the category of coalgebras in ${}_A\mathbf{BMod}_A(\mathbf{Ab}_*)$. By definition, we have an equivalence

$${}_A\mathbf{CoAlg}_A(\mathbf{Ab}_*) \simeq \mathbf{Alg}({}_A\mathbf{BMod}_A(\mathbf{Ab}_*)^{\text{op}})^{\text{op}}.$$

We denote by $\mathbf{LMod}_A(\mathbf{Ab}_*)$ the category of left A -modules and by $\mathbf{RMod}_A(\mathbf{Ab}_*)$ the category of right A -modules, respectively. Let $\Gamma \in {}_A\mathbf{CoAlg}_A(\mathbf{Ab}_*)$. A left Γ -comodule is defined to be a left A -module M equipped with a map

$$\psi : M \longrightarrow \Gamma \otimes_A M$$

in $\mathbf{LMod}_A(\mathbf{Ab}_*)$ satisfying the coassociativity and counit conditions. A right Γ -comodule is defined in the similar fashion. We denote by $\mathbf{LComod}_{(A,\Gamma)}(\mathbf{Ab}_*)$ the

category of left Γ -comodules and by $\mathbf{RComod}_{(A,\Gamma)}(\mathbf{Ab}_*)$ the category of right Γ -comodules, respectively.

The following lemma is obtained in the same way as in [29, Thm. A1.1.3 and Lem. A1.2.2].

Lemma 7. *If Γ is flat as a right A -module, then $\mathbf{LComod}_{(A,\Gamma)}(\mathbf{Ab}_*)$ is an abelian category with enough injectives.*

In the following of this subsection we assume that a coalgebra $\Gamma \in {}_A\mathbf{CoAlg}_A(\mathbf{Ab}_*)$ is flat as a right A -module and a left A -module. Hence we can do homological algebra in $\mathbf{LComod}_{(A,\Gamma)}(\mathbf{Ab}_*)$. We abbreviate $\mathbf{Hom}_{\mathbf{LComod}_{(A,\Gamma)}(\mathbf{Ab}_*)}(-, -)$ as $\mathbf{Hom}_\Gamma(-, -)$. For a left Γ -comodule M , we define

$$\mathbf{Ext}_\Gamma^i(M, -)$$

to be the i th right derived functor of

$$\mathbf{Hom}_\Gamma(M, -) : \mathbf{LComod}_{(A,\Gamma)}(\mathbf{Ab}_*) \longrightarrow \mathbf{Ab}_*.$$

For $M \in \mathbf{RComod}_{(A,\Gamma)}(\mathbf{Ab}_*)$ and $N \in \mathbf{LComod}_{(A,\Gamma)}(\mathbf{Ab}_*)$, we denote by $M \square_\Gamma N$ the cotensor product of M and N over Γ (see, for example, [29, Definition A1.1.4]). We consider the functor

$$M \square_\Gamma (-) : \mathbf{LComod}_{(A,\Gamma)}(\mathbf{Ab}_*) \longrightarrow \mathbf{Ab}_*.$$

We define

$$\mathbf{Cotor}_\Gamma^i(M, -)$$

to be the i th right derived functor of $M \square_\Gamma (-)$. Note that if M is flat as a right A -module, then $\mathbf{Cotor}_\Gamma^0(M, N) \cong M \square_\Gamma N$ since $M \square_\Gamma (-)$ is left exact in this case.

Let M be a left Γ -comodule that is finitely generated and projective as a left A -module. There is a right Γ -comodule structure on $\mathbf{Hom}_A(M, A)$ and we have a natural isomorphism

$$\mathbf{Hom}_\Gamma(M, N) \cong \mathbf{Hom}_A(M, A) \square_\Gamma N$$

for any $N \in \mathbf{LComod}_{(A,\Gamma)}(\mathbf{Ab}_*)$ (cf. [29, Lem. A1.1.6]). This implies that there is a natural isomorphism

$$\mathbf{Ext}_\Gamma^i(M, N) \cong \mathbf{Cotor}_\Gamma^i(\mathbf{Hom}_A(M, A), N)$$

for any $i \geq 0$. In particular, we have a natural isomorphism

$$\mathbf{Ext}_\Gamma^i(A, N) \cong \mathbf{Cotor}_\Gamma^i(A, N)$$

for any $N \in \mathbf{LComod}_{(A,\Gamma)}(\mathbf{Ab}_*)$ and $i \geq 0$.

For a right Γ -comodule M and a left Γ -comodule N , we have a cosimplicial object

$$C^\bullet(M, \Gamma, N)$$

in Ab_* obtained by the cobar construction. In particular, we have

$$C^r(M, \Gamma, N) = M \otimes_A \Gamma^{\otimes_A r} \otimes_A N$$

for $r \geq 0$. The cobar complex $C^*(M, \Gamma, N)$ is the associated cochain complex. The normalized cobar complex $\overline{C}^*(M, \Gamma, N)$ is a subcomplex of $C^*(M, \Gamma, N)$ that is given by

$$\overline{C}^r(M, \Gamma, N) = M \otimes_A \overline{\Gamma}^{\otimes_A r} \otimes_A N,$$

for $r \geq 0$, where $\overline{\Gamma} = \ker \varepsilon$.

We say that a left Γ -comodule N is relatively injective if N is a direct summand of $\Gamma \otimes_A N'$ as a left Γ -comodule for some left A -module N' . For a left Γ -comodule N , the map $\psi : N \rightarrow \Gamma \otimes_A N$ induces an augmentation $N \rightarrow C^*(\Gamma, \Gamma, N)$. This gives a resolution of N in $\text{LComod}_{(A, \Gamma)}(\text{Ab}_*)$ by relative injectives. Note that the resolution is split in $\text{LMod}_A(\text{Ab}_*)$. The splitting is given by

$$\varepsilon \otimes 1^{\otimes r} \otimes 1 : \Gamma \otimes_A \Gamma^{\otimes_A r} \otimes_A N \longrightarrow A \otimes_A \Gamma^{\otimes_A r} \otimes_A N \cong \Gamma \otimes_A \Gamma^{\otimes_A (r-1)} \otimes_A N.$$

Similarly, we have a resolution $N \rightarrow \overline{C}^*(\Gamma, \Gamma, N)$ of N in $\text{LComod}_{(A, \Gamma)}(\text{Ab}_*)$ by relative injectives that is split in $\text{LMod}_A(\text{Ab}_*)$. By the proof of [29, Lemma A1.2.9], the cobar complex $C^*(\Gamma, \Gamma, N)$ is cochain homotopy equivalent to the normalized cobar complex $\overline{C}^*(\Gamma, \Gamma, N)$. Since $C^*(M, \Gamma, N) \cong M \square_\Gamma C^*(\Gamma, \Gamma, N)$ and $\overline{C}^*(M, \Gamma, N) \cong M \square_\Gamma \overline{C}^*(\Gamma, \Gamma, N)$, this implies that

$$H^*(C^*(M, \Gamma, N)) \cong H^*(\overline{C}^*(M, \Gamma, N))$$

for any $M \in \text{RComod}_{(A, \Gamma)}(\text{Ab}_*)$ and $N \in \text{LComod}_{(A, \Gamma)}(\text{Ab}_*)$.

The following proposition is obtained in the same way as in [29, Cor. A1.2.12].

Proposition 9. *If M is flat as a right A -module, then*

$$H^*(C^*(M, \Gamma, N)) \cong \text{Cotor}_\Gamma^*(M, N).$$

In particular, we have

$$H^*(C^*(A, \Gamma, N)) \cong \text{Ext}_\Gamma^*(A, N).$$

5.2 Bousfield-Kan spectral sequences

In this subsection we work in the quasi-category of spectra Sp and study the Bousfield-Kan spectral sequence abutting to the homotopy groups of cotensor products of comodules.

Note that Sp is a presentable stable symmetric monoidal category in which the tensor product commutes with all colimits separately in each variable. We use \otimes for

the tensor product in \mathbf{Sp} instead of \wedge . We denote by \mathbb{S} the sphere spectrum that is the monoidal unit.

We would like to compute the homotopy groups of cotensor products of comodules. Since a cotensor product of comodules is a limit of a cosimplicial object, we have the Bousfield-Kan spectral sequence abutting to the homotopy groups of the cotensor product.

First, we recall the Bousfield-Kan spectral sequence associated to a cosimplicial object in \mathbf{Sp} . Let $X^\bullet : N(\Delta) \rightarrow \mathbf{Sp}$ be a cosimplicial object in \mathbf{Sp} . Since the quasi-category \mathbf{Sp} of spectra is the underlying quasi-category of the combinatorial simplicial model category $\Sigma\mathbf{Sp}$ of symmetric spectra, we can take a cosimplicial object $Y^\bullet : \Delta \rightarrow \Sigma\mathbf{Sp}^\circ$ such that $N(Y^\bullet) \simeq X^\bullet$ by [20, Prop. 4.2.4.4.], where $\Sigma\mathbf{Sp}^\circ$ is the simplicial full subcategory of $\Sigma\mathbf{Sp}$ consisting of objects that are both fibrant and cofibrant. Then the limit $\lim_{N(\Delta)} X^\bullet$ in \mathbf{Sp} is represented by the homotopy limit $\mathrm{holim}_\Delta Y^\bullet$.

We recall that $\mathrm{Tot}_r(X^\bullet)$ is defined to be the limit of $X^\bullet|_{N(\Delta^{\leq r})}$ in \mathbf{Sp} for $r \geq 0$, where $\Delta^{\leq r}$ is the full subcategory of Δ spanned by $\{[0], [1], \dots, [r]\}$. The inclusion $\Delta^{\leq r} \hookrightarrow \Delta^{\leq r+1}$ induces a map $\mathrm{Tot}_{r+1}(X^\bullet) \rightarrow \mathrm{Tot}_r(X^\bullet)$ for $r \geq 0$. We have a tower $\{\mathrm{Tot}_r(X^\bullet)\}_{r \geq 0}$ and the limit of the tower is equivalent to $\mathrm{Tot}(X^\bullet)$:

$$\mathrm{Tot}(X^\bullet) \simeq \lim_r \mathrm{Tot}_r(X^\bullet).$$

Let $F_r(X^\bullet)$ be the fiber of the map $\mathrm{Tot}_r(X^\bullet) \rightarrow \mathrm{Tot}_{r-1}(X^\bullet)$ for $r \geq 0$, where $\mathrm{Tot}_{-1}(X^\bullet) = 0$. Associated to the tower $\{\mathrm{Tot}_r(X^\bullet)\}_{r \geq 0}$, by applying the homotopy groups, we obtain the Bousfield-Kan spectral sequence

$$E_1^{s,t} \cong \pi_{t-s} F_s(X^\bullet) \implies \pi_{t-s} \mathrm{Tot}(X^\bullet)$$

(see [4, Ch. IX, §4]). We can identify the E_2 -page of the spectral sequence with the cohomotopy groups of the cosimplicial graded abelian group $\pi_*(X^\bullet)$:

$$E_2^{s,t} \cong \pi^s \pi_t(X^\bullet)$$

(see [4, Ch. X, §7]).

Next, we construct a spectral sequence that computes the homotopy groups of cotensor products of comodules. Let A be an algebra object of \mathbf{Sp} and Γ a coalgebra object of ${}_A\mathbf{BMod}_A(\mathbf{Sp})$. Recall that the cotensor product $M \square_\Gamma N$ is defined to be the limit of the cosimplicial object $C^\bullet(M, \Gamma, N)$ for a right Γ -comodule M and a left Γ -comodule N . Hence we obtain the Bousfield-Kan spectral sequence abutting to the homotopy groups of the cotensor product $M \square_\Gamma N$:

$$E_2^{s,t} \implies \pi_{t-s}(M \square_\Gamma N),$$

where the E_2 -page is given by

$$E_2^{s,t} \cong \pi^s \pi_t C^\bullet(M, \Gamma, N).$$

For a spectrum $X \in \mathbf{Sp}$, we write X_* for the homotopy groups π_*X for simplicity. Now we suppose that Γ_* is flat as a left A_* -module and a right A_* -module. Since

$$C^r(M, \Gamma, N) \simeq M \otimes_A \overbrace{\Gamma \otimes_A \cdots \otimes_A \Gamma}^r \otimes_A N \text{ for all } r \geq 0, \text{ we see that}$$

$$\pi_* C^\bullet(M, \Gamma, N) \cong C^\bullet(M_*, \Gamma_*, N_*)$$

if M_* is a flat right A_* -module or N_* is a flat left A_* -module.

Proposition 10. *If M_* is a flat right A_* -module or N_* is a flat left A_* -module, then we have the Bousfield-Kan spectral sequence abutting to the homotopy groups of the cotensor product $M \square_\Gamma N$:*

$$E_2^{s,t} \implies \pi_{t-s}(M \square_\Gamma N).$$

The E_2 -page of the spectral sequence is given by

$$E_2^{s,t} \cong \text{Cotor}_{\Gamma_*}^s(M_*, N_*)_t.$$

Now we regard A as a right A -module and suppose A is a right Γ -comodule via η_R . In this case A_* is a right Γ_* -comodule via η_{R*} . Then we can regard A_* as a left Γ_* -comodule by using the isomorphism $A_* \cong \text{Hom}_{A_*}(A_*, A_*)$ of left A_* -modules, where $\text{Hom}_{A_*}(A_*, A_*)$ is the graded abelian group of graded homomorphisms of right A_* -modules. Hence we can form $\text{Ext}_{\Gamma_*}^s(A_*, N_*)$ for any left Γ -comodule N .

We consider the Bousfield-Kan spectral sequence associated to $C^\bullet(A, \Gamma, N)$. Note that the limit of $C^\bullet(A, \Gamma, N)$ is $P(N) = A \square_\Gamma N$.

Corollary 2. *We assume that the right A -module A is a right Γ -comodule via η_R . Then we have the Bousfield-Kan spectral sequence abutting to the homotopy groups of $P(N)$:*

$$E_2^{s,t} \implies \pi_{t-s}P(N),$$

where the E_2 -page is given by

$$E_2^{s,t} \cong \text{Ext}_{\Gamma_*}^s(A_*, N_*)_t.$$

In the following of this subsection we study the relationship between Bousfield-Kan spectral sequences and Adams spectral sequences.

For an \mathbb{S} -algebra A , we have the coalgebra $A \otimes A$ in ${}_A\mathbf{BMod}_A(\mathbf{Sp})$. We write $\Gamma(A) = (A, A \otimes A)$ for simplicity and we call $A \otimes A$ -comodules $\Gamma(A)$ -comodules interchangeably. We can regard A as a left $\Gamma(A)$ -comodule via $\eta_L : A \simeq A \otimes \mathbb{S} \xrightarrow{\text{id}_A \otimes u} A \otimes A$ and as a right $\Gamma(A)$ -comodule via $\eta_R : A \simeq \mathbb{S} \otimes A \xrightarrow{u \otimes \text{id}_A} A \otimes A$, where $u : \mathbb{S} \rightarrow A$ is the unit map.

By Theorem 5, we have a left $\Gamma(A)$ -comodule $A \otimes X$ for any $X \in \mathbf{Sp}$. We consider the cobar construction

$$C^\bullet(A, A \otimes A, A \otimes X),$$

where $X \in \mathbf{Sp}$. Note that we have a coaugmentation $X \rightarrow C^\bullet(A, A \otimes A, A \otimes X)$, which is given by $X \simeq \mathbb{S} \otimes X \xrightarrow{u \otimes \text{id}_X} A \otimes X \simeq C^0(A, A \otimes A, A \otimes X)$. This induces a map

$$X \rightarrow P(A \otimes X) = \lim_{N(\Delta)} C^\bullet(A, A \otimes A, A \otimes X).$$

We have an equivalence $C^\bullet(A, A \otimes A, A \otimes X) \simeq C^\bullet(A, A \otimes A, A) \otimes X$. We see that the cobar construction $C^\bullet(A, A \otimes A, A)$ is the Amitsur complex in \mathbf{Sp} given by

$$C^r(A, A \otimes A, A) \simeq \overbrace{A \otimes \cdots \otimes A}^{r+1}$$

for any $r \geq 0$ with the usual structure maps. The Bousfield-Kan spectral sequence of the cobar construction $C^\bullet(A, A \otimes A, A \otimes X)$ is related to the A -Adams spectral sequence of X . Although this may be well-known to experts, we briefly review this relation for the reader's convenience (see, for example, [24, §2.1]).

The coaugmented cosimplicial object $X \rightarrow C^\bullet(A, A \otimes A, A \otimes X)$ induces a tower

$$\{\text{Tot}_r C^\bullet(A, A \otimes A, A \otimes X)\}_{r \geq 0},$$

and a map of towers $c(X) \rightarrow \{\text{Tot}_r C^\bullet(A, A \otimes A, A \otimes X)\}_{r \geq 0}$ for any $X \in \mathbf{Sp}$. This tower is related to the A -Adams tower of X .

Let \bar{A} be the fiber of the unit map $u : \mathbb{S} \rightarrow A$. We have a canonical map $\bar{A} \rightarrow \mathbb{S}$. For $r \geq 0$, we set

$$T_r(A, X) = \overbrace{\bar{A} \otimes \cdots \otimes \bar{A}}^r \otimes X,$$

where we understand $\bar{A}^{\otimes 0} = \mathbb{S}$. Using the canonical map $\bar{A} \rightarrow \mathbb{S}$, we define a map $T_{r+1}(A, X) \rightarrow T_r(A, X)$ for $r \geq 0$ by

$$T_{r+1}(A, X) \simeq \bar{A} \otimes \bar{A}^{\otimes r} \otimes X \rightarrow \mathbb{S} \otimes \bar{A}^{\otimes r} \otimes X \simeq T_r(A, X).$$

With these maps, we obtain a tower $\{T_r(A, X)\}_{r \geq 0}$ and a map $\{T_r(A, X)\}_{r \geq 0} \rightarrow c(X)$ of towers.

Let $G_r(A, X)$ be the cofiber of the map $T_{r+1}(A, X) \rightarrow T_r(A, X)$ for $r \geq 0$. Associate to the tower $\{T_r(A, X)\}_{r \geq 0}$, by applying the homotopy groups, we obtain the A -Adams spectral sequence of X . The E_1 -page of the spectral sequence is given by

$$E_1^{s,t} \cong \pi_{t-s} G_s(A, X)$$

(see, for example, [29, Ch. 2.2]).

We set $C^\bullet = C^\bullet(A, A \otimes A, A \otimes X)$. By [24, Prop. 2.14], the cofiber of the map $T_{r+1}(A, X) \rightarrow X$ is equivalent to $\text{Tot}_r(C^\bullet)$ for all $r \geq 0$. Hence we obtain a natural cofiber sequence of towers

$$\{T_{r+1}(A, X)\}_{r \geq 0} \rightarrow c(X) \rightarrow \{\text{Tot}_r(C^\bullet)\}_{r \geq 0}.$$

In particular, we see that $G_r(A, X)$ is equivalent to the fiber $F_r(C^\bullet)$ of the map $\text{Tot}_r(C^\bullet) \rightarrow \text{Tot}_{r-1}(C^\bullet)$. Comparing the spectral sequences, we see that the A -Adams spectral sequence of X coincides with the Bousfield-Kan spectral sequence associated to the cobar construction $C^\bullet(A, A \otimes A, A \otimes X)$.

We recall that the map $X \rightarrow P(A \otimes X)$ is an A -nilpotent completion in $\text{Ho}(\text{Sp})$ in the sense of Bousfield [3], where $\text{Ho}(\text{Sp})$ is the stable homotopy category of spectra.

Let R be a ring spectrum in $\text{Ho}(\text{Sp})$. A spectrum W is said to be R -nilpotent if W lies in the thick ideal of $\text{Ho}(\text{Sp})$ generated by R . An R -nilpotent resolution of a spectrum Z is a tower $\{W_r\}_{r \geq 0}$ equipped with a map of towers $c(Z) \rightarrow \{W_r\}_{r \geq 0}$ in $\text{Ho}(\text{Sp})$ such that W_r is R -nilpotent for all $r \geq 0$ and the map

$$\text{colim}_r \text{Hom}_{\text{Ho}(\text{Sp})}(W_r, N) \rightarrow \text{Hom}_{\text{Ho}(\text{Sp})}(Z, N)$$

is an isomorphism for any R -nilpotent spectrum N . An R -nilpotent completion of Z is defined to be the map $Z \rightarrow \text{holim}_r W_r$ for an R -nilpotent resolution $\{W_r\}_{r \geq 0}$ of Z .

We shall show that the tower $\{\text{Tot}_r(C^\bullet)\}_{r \geq 0}$ is an A -nilpotent resolution of X , where $C^\bullet = C^\bullet(A, A \otimes A, A \otimes X)$. For any $r \geq 0$, the fiber $F_r(C^\bullet)$ of the map $\text{Tot}_r(C^\bullet) \rightarrow \text{Tot}_{r-1}(C^\bullet)$ is equivalent to $G_r(A, X)$. Since $G_r(A, X) \simeq A \otimes \bar{A}^{\otimes r} \otimes X$ is a left A -module, $G_r(A, X)$ is A -nilpotent for all $r \geq 0$. By induction on r and the fact that $\text{Tot}_0(C^\bullet) = A \otimes X$, we see that $\text{Tot}_r(C^\bullet)$ is A -nilpotent for all $r \geq 0$.

Recall that the fiber of the map $X \rightarrow \text{Tot}_r(C^\bullet)$ is equivalent to $T_{r+1}(A, X)$ for all $r \geq 0$. The map $\bar{A} \rightarrow \mathbb{S}$ is null in $\text{Ho}(\text{Sp})$ after tensoring with A since the unit map $\mathbb{S} \rightarrow A$ has a left inverse after tensoring with A . Hence we see that the map $T_{r+1}(A, X) \rightarrow T_r(A, X)$ is null in $\text{Ho}(\text{Sp})$ after tensoring with A . This implies that the map $\text{colim}_r \text{Hom}_{\text{Ho}(\text{Sp})}(\text{Tot}_r(C^\bullet), A \otimes Y) \rightarrow \text{Hom}_{\text{Ho}(\text{Sp})}(X, A \otimes Y)$ is an isomorphism for any spectrum Y . Since the class of A -nilpotent spectra coincides with the thick subcategory generated by the class $\{A \otimes Z \mid Z \in \text{Sp}\}$, we see that the map $\text{colim}_r \text{Hom}_{\text{Ho}(\text{Sp})}(\text{Tot}_r(C^\bullet), N) \rightarrow \text{Hom}_{\text{Ho}(\text{Sp})}(X, N)$ is an isomorphism for any A -nilpotent spectrum N .

Therefore, the tower $\{\text{Tot}_r(C^\bullet)\}_{r \geq 0}$ is an A -nilpotent resolution of X . Since $P(A \otimes X) \simeq \lim_r \text{Tot}_r(C^\bullet)$, we see that the map $X \rightarrow P(A \otimes X)$ is an A -nilpotent completion in $\text{Ho}(\text{Sp})$.

5.3 Complex oriented spectra

In this subsection we study quasi-categories of comodules over coalgebras associated to Landweber exact \mathbb{S} -algebras. We show that the quasi-category of comodules over the coalgebra associated to a Landweber exact \mathbb{S} -algebra depends only on the height of the underlying MU_* -algebra.

Let MU be the complex cobordism spectrum. The coefficient ring of MU is a polynomial ring over the ring \mathbb{Z} of integers with infinite many variables

$$\pi_* MU = \mathbb{Z}[x_1, x_2, \dots]$$

with degree $|x_i| = 2i$ for $i \geq 1$. We assume that the Chern numbers of x_{p^n-1} are all divisible by p for all positive integers n and all prime numbers p . In this case the ideals $I_{p,n} = (p, x_{p-1}, \dots, x_{p^n-1})$ are invariant and independent of the choice of generators. We set $I_{p,0} = (0)$ and $I_{p,\infty} = \bigcup_{n \geq 0} I_{p,n}$. The ideals $I_{p,n}$ for $0 \leq n \leq \infty$ and all primes p are the only invariant prime ideals in MU_* (see [18]).

For a graded commutative MU_* -algebra R_* , we say that R_* is Landweber exact if $p, x_{p-1}, \dots, x_{p^n-1}, \dots$ is a regular sequence in R_* for all prime numbers p .

If $E^*(-)$ is a complex oriented cohomology theory represented by a spectrum E , then there is a ring spectrum map $f : MU \rightarrow E$ in the stable homotopy category $\text{Ho}(\text{Sp})$ of spectra. We say E is Landweber exact if E_* is a graded commutative ring and Landweber exact via the graded ring homomorphism $f_* : MU_* \rightarrow E_*$.

We consider an \mathbb{S} -algebra that is Landweber exact.

Definition 1. We say that A is Landweber exact \mathbb{S} -algebra if A is an \mathbb{S} -algebra spectrum equipped with a map $f : MU \rightarrow A$ of ring spectra in $\text{Ho}(\text{Sp})$ such that A_* is a graded commutative ring and Landweber exact via the graded ring homomorphism $f_* : MU_* \rightarrow A_*$.

Let p be a prime number and let $\mathbb{S}_{(p)}$ be the localization of the sphere spectrum \mathbb{S} at p . We can consider a Landweber exact $\mathbb{S}_{(p)}$ -algebra in the same way. If A is a Landweber exact \mathbb{S} -algebra, then the localization $A_{(p)}$ is a Landweber exact $\mathbb{S}_{(p)}$ -algebra at any prime number p .

Example 1. For any prime number p and any positive integer n , the Johnson-Wilson spectrum $E(n)$ at p is a complex oriented Landweber exact spectrum. By [1, Proposition 4.1], $E(n)$ admits an $MU_{(p)}$ -algebra spectrum structure. Hence, in particular, $E(n)$ is a Landweber exact $\mathbb{S}_{(p)}$ -algebra.

Definition 2. For a Landweber exact graded commutative ring A_* , we denote by $\text{ht}_p A_*$ the height of A_* at a prime p in the sense of [17, Definition 7.2], that is, the largest number n such that $A_*/I_{p,n}$ is nonzero, or ∞ if $A_*/I_{p,n}$ is nonzero for all n . For a Landweber exact \mathbb{S} -algebra A , we denote by $\text{ht}_p A$ the height of A_* at p .

For Landweber exact \mathbb{S} -algebras E and F , we have an isomorphism

$$F_*(E) \cong F_* \otimes_{MU_*} MU_*(MU) \otimes_{MU_*} E_*.$$

By abuse of notation, for graded commutative Landweber exact MU_* -algebras A_* and B_* , we set $B_*(A) = B_* \otimes_{MU_*} MU_*(MU) \otimes_{MU_*} A_*$. We denote by $\Gamma(A_*)$ the pair $(A_*, A_*(A))$, which forms a graded Hopf algebroid (see [29, Appendix A.1]). We can consider the categories of graded $\Gamma(A_*)$ -comodules $\text{LComod}_{\Gamma(A_*)}(\text{Ab}_*)$ which is an abelian category since $\Gamma(A_*)$ is a flat Hopf algebroid.

The canonical map $MU_*(MU) \rightarrow A_*(A)$ induces a map of graded Hopf algebroids $\Phi(A) : \Gamma(MU_*) \rightarrow \Gamma(A_*)$. We consider the functor

$$\Phi(A)_* : \text{LComod}_{\Gamma(MU_*)}(\text{Ab}_*) \longrightarrow \text{LComod}_{\Gamma(A_*)}(\text{Ab}_*)$$

given by $\Phi(A)_*(M_*) = A_* \otimes_{MU_*} M_*$ for $M_* \in \mathrm{LComod}_{\Gamma(MU_*)}(\mathrm{Ab}_*)$. The functor $\Phi(A)_*$ has the right adjoint $\Phi(A)^* : \mathrm{LComod}_{\Gamma(A_*)}(\mathrm{Ab}_*) \rightarrow \mathrm{LComod}_{\Gamma(MU_*)}(\mathrm{Ab}_*)$ given by

$$\Phi(A)^*(N_*) = MU_*(A) \square_{A_*(A)} N_*$$

for $N_* \in \mathrm{LComod}_{\Gamma(A_*)}(\mathrm{Ab}_*)$ (see [17, Lem. 2.4 and Remark after its proof]). Let \mathcal{T}_{A_*} be the class of all graded $\Gamma(MU_*)$ -comodules M_* such that $A_* \otimes_{MU_*} M_*$ is trivial. By [17, Thm. 2.5], the adjoint pair $(\Phi(A)_*, \Phi(A)^*)$ induces an adjoint equivalence of categories between $\mathrm{LComod}_{\Gamma(A_*)}(\mathrm{Ab}_*)$ and the localization of $\mathrm{LComod}_{\Gamma(MU_*)}$ with respect to \mathcal{T}_{A_*} .

Let A_* and B_* be graded commutative Landweber exact MU_* -algebras. We recall that $B_*(A) = B_* \otimes_{MU_*} MU_*(MU) \otimes_{MU_*} A_*$. Note that $B_*(A)$ is a graded left $\Gamma(B_*)$ -comodule and a graded right $\Gamma(A_*)$ -comodule. We can define a functor

$$G_{B_*, A_*} : \mathrm{LComod}_{\Gamma(A_*)}(\mathrm{Ab}_*) \longrightarrow \mathrm{LComod}_{\Gamma(B_*)}(\mathrm{Ab}_*)$$

which assigns to a graded left $\Gamma(A_*)$ -comodule M_* the graded left $\Gamma(B_*)$ -comodule $G_{B_*, A_*}(M_*)$ given by

$$G_{B_*, A_*}(M_*) = B_*(A) \square_{A_*(A)} M_*.$$

Lemma 8. *If $\mathrm{ht}_p A_* = \mathrm{ht}_p B_*$ for all primes p , then the functor G_{B_*, A_*} gives an equivalence of categories between $\mathrm{LComod}_{\Gamma(A_*)}(\mathrm{Ab}_*)$ and $\mathrm{LComod}_{\Gamma(B_*)}(\mathrm{Ab}_*)$.*

Proof. Note that the functor G_{B_*, A_*} is the composition $\Phi(B)_* \Phi(A)^*$. The functor $\Phi(A)^*$ induces an equivalence of categories from $\mathrm{LComod}_{\Gamma(A_*)}(\mathrm{Ab}_*)$ to the localization of $\mathrm{LComod}_{\Gamma(MU_*)}(\mathrm{Ab}_*)$ with respect to \mathcal{T}_{A_*} and $\Phi(B)_*$ induces an equivalence of categories from the localization of $\mathrm{LComod}_{\Gamma(MU_*)}(\mathrm{Ab}_*)$ with respect to \mathcal{T}_{B_*} to $\mathrm{LComod}_{\Gamma(B_*)}(\mathrm{Ab}_*)$. By [17, Thm. 7.3], the assumption that A_* and B_* have the same heights for all p implies that $\mathcal{T}_{A_*} = \mathcal{T}_{B_*}$. Hence we see that G_{B_*, A_*} gives an equivalence of categories. \square

Lemma 9. *Let A, B, C be Landweber exact \mathbb{S} -algebras. We assume that $\mathrm{ht}_p A = \mathrm{ht}_p B = \mathrm{ht}_p C$ for all primes p . Then, for any $\Gamma(A)$ -comodule M , the canonical map*

$$C \otimes ((B \otimes A) \square_{A \otimes A} M) \longrightarrow (C \otimes B \otimes A) \square_{A \otimes A} M$$

is an equivalence.

Proof. Let R be a Landweber exact \mathbb{S} -algebra which has the same height at all p as A . First, we consider the homotopy groups of the cotensor product $(R \otimes A) \square_{A \otimes A} M$. The Bousfield-Kan spectral sequence abutting to the homotopy groups of $(R \otimes A) \square_{A \otimes A} M$ has the E_2 -page given by

$$E_2^{s,t} \cong \mathrm{Cotor}_{A_*(A)}^s(R_*(A), M_*)_t.$$

We have

$$H^s(C(A_*(A), A_*(A), M_*)) = 0$$

for $s > 0$. Note that the cobar complex $C^*(A_*(A), A_*(A), M_*)$ is a cochain complex in the abelian category $\text{LComod}_{\Gamma(A_*)}(\text{Ab}_*)$. Applying the functor G_{R_*, A_*} to $C^*(C(A_*(A), A_*(A), M_*))$, we obtain

$$H^s(C(R_*(A), A_*(A), M_*)) = 0$$

for $s > 0$ by Lemma 8. Hence the Bousfield-Kan spectral sequence abutting to the homotopy groups of $(R \otimes A) \square_{A \otimes A} M$ collapses from the E_2 -page and we obtain an isomorphism

$$\pi_*((R \otimes A) \square_{A \otimes A} M) \cong R_*(A) \square_{A_*(A)} M_*.$$

In particular, since $C \otimes B$ is Landweber exact through the map $MU \rightarrow B \rightarrow C \otimes B$ in $\text{Ho}(\text{Sp})$, we obtain an isomorphism

$$\pi_*((C \otimes B \otimes A) \square_{A \otimes A} M) \cong (C \otimes B)_*(A) \square_{A_*(A)} M_*.$$

Since we have an isomorphism

$$(C \otimes B)_*(A) \cong C_*(B) \otimes_{B_*} B_*(A),$$

we obtain an isomorphism

$$\pi_*((C \otimes B \otimes A) \square_{A \otimes A} M) \cong C_*(B) \otimes_{B_*} (B_*(A) \square_{A_*(A)} M_*).$$

On the other hand, we have isomorphisms

$$\pi_*(C \otimes ((B \otimes A) \square_{A \otimes A} M)) \cong C_*(B) \otimes_{B_*} \pi_*((B \otimes A) \square_{A \otimes A} M).$$

and

$$\pi_*((B \otimes A) \square_{A \otimes A} M) \cong B_*(A) \square_{A_*(A)} M_*.$$

Hence we see that the canonical map $C \otimes (B \otimes A) \square_{A \otimes A} M \rightarrow (C \otimes B \otimes A) \square_{A \otimes A} M$ induces an isomorphism of homotopy groups. This completes the proof. \square

Corollary 3. *Let A and B be Landweber exact \mathbb{S} -algebras. We assume that $\text{ht}_p A = \text{ht}_p B$ for all primes p . For any left $\Gamma(A)$ -comodule M , the left $\Gamma(B)$ -comodule structure on B induces a left $\Gamma(B)$ -comodule structure on $(B \otimes A) \square_{A \otimes A} M$.*

Proof. For $r > 0$, $B^{\otimes r}$ is a Landweber exact \mathbb{S} -algebra and has the same height at all p as B . By Lemma 9, the canonical map $B^{\otimes r} \otimes ((B \otimes A) \square_{A \otimes A} M) \rightarrow (B^{\otimes r} \otimes B \otimes A) \square_{A \otimes A} M$ is an equivalence for all $r > 0$. Applying Lemma 4 for the simplicial object $B_\bullet(B \otimes A, A \otimes A, M)$ in $\text{LMod}_{B \otimes B}(\text{LMod}_B(\text{Sp})^{\text{op}})$, we see that the cosimplicial object $C^\bullet(B \otimes A, A \otimes A, M)$ has a limit in $\text{LComod}_{\Gamma(B)}(\text{Sp})$ and the forgetful functor $\text{LComod}_{\Gamma(B)}(\text{Sp}) \rightarrow \text{LMod}_B(\text{Sp})$ preserves the limit. \square

Using Corollary 3, we can define a functor

$$F_{B,A} : \text{Comod}_{\Gamma(A)}(\text{Sp}) \longrightarrow \text{Comod}_{\Gamma(B)}(\text{Sp})$$

by assigning to $M \in \text{LComod}_{\Gamma(A)}(\text{Sp})$ the $\Gamma(B)$ -comodule $F_{B,A}(M)$ given by

$$F_{B,A}(M) = (B \otimes A) \square_{A \otimes A} M.$$

Theorem 8. *Let A and B be Landweber exact \mathbb{S} -algebras. We assume that A and B have the same height at all p . Then the functor $F_{B,A}$ gives an equivalence of quasi-categories*

$$\mathrm{LComod}_{\Gamma(A)}(\mathrm{Sp}) \simeq \mathrm{LComod}_{\Gamma(B)}(\mathrm{Sp}).$$

Proof. For any left $\Gamma(A)$ -comodule M , we have

$$\pi_* F_{B,A}(M) \cong B_*(A) \square_{A_*(A)} M_*.$$

Since the functor $G_{B_*,A_*} = B_*(A) \square_{A_*(A)} (-)$ gives an equivalence of categories by Lemma 8, we see that the functor $F_{B,A}$ gives an equivalence of quasi-categories between $\mathrm{LComod}_{\Gamma(A)}(\mathrm{Sp})$ and $\mathrm{LComod}_{\Gamma(B)}(\mathrm{Sp})$. \square

Proposition 11. *Let A and B be Landweber exact \mathbb{S} -algebras. We assume that $\mathrm{ht}_p A = \mathrm{ht}_p B$ for all primes p . Then the following diagram is commutative*

$$\begin{array}{ccc} & \mathrm{Sp} & \\ A \otimes (-) \swarrow & & \searrow B \otimes (-) \\ \mathrm{LComod}_{\Gamma(A)} & \xrightarrow{F_{B,A}} & \mathrm{LComod}_{\Gamma(B)}. \end{array}$$

Proof. Since $B \otimes \mathbb{S} \simeq B \otimes_{\mathbb{S}} \mathbb{S} \otimes \mathbb{S}$ is an extended right $\Gamma(\mathbb{S})$ -comodule, we have a natural equivalence

$$B \otimes X \simeq (B \otimes \mathbb{S}) \square_{\mathbb{S} \otimes \mathbb{S}} (\mathbb{S} \otimes X)$$

for any $X \in \mathrm{Sp}$. The unit map $\mathbb{S} \rightarrow A$ induces a natural map

$$(B \otimes \mathbb{S}) \square_{\mathbb{S} \otimes \mathbb{S}} (\mathbb{S} \otimes X) \longrightarrow (B \otimes A) \square_{A \otimes A} (A \otimes X)$$

and hence we obtain a natural map

$$f : B \otimes X \longrightarrow (B \otimes A) \square_{A \otimes A} (A \otimes X).$$

Note that f is a map of left $\Gamma(B)$ -comodule. Since

$$\pi_*((B \otimes A) \square_{A \otimes A} (A \otimes X)) \cong B_*(A) \square_{A_*(A)} A_*(X),$$

we see that f induces an isomorphism of homotopy groups. This completes the proof. \square

5.4 The $E(n)$ -local category

In this subsection we study the quasi-category of $E(n)$ -local spectra, where $E(n)$ is the n th Johnson-Wilson spectrum at a prime p . We show that the quasi-category of $E(n)$ -local spectra is equivalent to the quasi-category of comodules over the coalgebra $A \otimes A$ for any Landweber exact $\mathbb{S}_{(p)}$ -algebra of height n at p .

In this subsection we fix a non-negative integer n and a prime number p . Let $L_n\mathrm{Sp}$ be the Bousfield localization of the quasi-category of spectra with respect to the n th Johnson-Wilson spectrum $E(n)$ at p . We denote by $L_n : \mathrm{Sp} \rightarrow \mathrm{Sp}$ the associated localization functor. The quasi-category $L_n\mathrm{Sp}$ is a stable homotopy theory with the tensor product in Sp and the unit $L_n\mathbb{S}$, where $L_n\mathbb{S}$ is the $E(n)$ -localization of the sphere spectrum \mathbb{S} .

We recall that $E(n)$ is a Landweber exact $\mathbb{S}_{(p)}$ -algebra with height $\mathrm{ht}_p E(n) = n$. For simplicity, we set $\Gamma(n) = (E(n), E(n) \otimes E(n))$. The functor

$$E(n) \otimes (-) : \mathrm{Sp} \rightarrow \mathrm{LComod}_{\Gamma(n)}(\mathrm{Sp})$$

factors through $L_n\mathrm{Sp}$ and we obtain a functor

$$E(n) \otimes (-) : L_n\mathrm{Sp} \longrightarrow \mathrm{LComod}_{\Gamma(n)}(\mathrm{Sp}).$$

Since any $N \in \mathrm{LMod}_{E(n)}(\mathrm{Sp})$ is $E(n)$ -local, we see that $P(M)$ lies in $L_n\mathrm{Sp}$ for any $M \in \mathrm{LComod}_{\Gamma(n)}(\mathrm{Sp})$, and we obtain an adjunction of functors

$$E(n) \otimes (-) : L_n\mathrm{Sp} \rightleftarrows \mathrm{LComod}_{\Gamma(n)}(\mathrm{Sp}) : P.$$

Proposition 12. *The pair of functors*

$$E(n) \otimes (-) : L_n\mathrm{Sp} \rightleftarrows \mathrm{LComod}_{\Gamma(n)}(\mathrm{Sp}) : P$$

is an adjoint equivalence.

Proof. By [14, Theorem 5.3], the unit $L_n\mathbb{S}$ is $E(n)$ -nilpotent (see, also, [30, Chapter 8]). By Theorem 7, we obtain the proposition. \square

Let A be a Landweber exact $\mathbb{S}_{(p)}$ -algebra with $\mathrm{ht}_p A = n$. Since A is Bousfield equivalent to $E(n)$ by [12, Corollary 1.12], we have a canonical equivalence of stable homotopy theories

$$L_A\mathrm{Sp} \simeq L_n\mathrm{Sp},$$

where $L_A\mathrm{Sp}$ is the Bousfield localization of Sp with respect to A . In the same way as $E(n)$, we have an adjunction of functors

$$A \otimes (-) : L_A\mathrm{Sp} \rightleftarrows \mathrm{LComod}_{\Gamma(A)}(\mathrm{Sp}) : P.$$

Recall that we have the functor

$$F_{A,E(n)} : \mathrm{LComod}_{\Gamma(n)}(\mathrm{Sp}) \rightarrow \mathrm{LComod}_{\Gamma(A)}(\mathrm{Sp})$$

given by

$$F_{A,E(n)}(M) = (A \otimes E(n)) \square_{E(n) \otimes E(n)} M$$

for $M \in \mathrm{LComod}_{\Gamma(n)}(\mathrm{Sp})$. By Proposition 11, we see that there is a commutative diagram of quasi-categories

$$\begin{array}{ccc} L_n \mathrm{Sp} & \xrightarrow{E(n) \otimes (-)} & \mathrm{LComod}_{\Gamma(n)}(\mathrm{Sp}) \\ \downarrow & & \downarrow F_{A,E(n)} \\ L_A \mathrm{Sp} & \xrightarrow{A \otimes (-)} & \mathrm{LComod}_{\Gamma(A)}(\mathrm{Sp}), \end{array}$$

where the left vertical arrow is the canonical equivalence.

Theorem 9. *If A is a Landweber exact $\mathbb{S}_{(p)}$ -algebra of height n at p , then the pair of functors*

$$A \otimes (-) : L_A \mathrm{Sp} \rightleftarrows \mathrm{Comod}_{A \otimes A}(\mathrm{Sp}) : P$$

is an adjoint equivalence.

Proof. The theorem follows from Theorem 8 and Proposition 12. \square

5.5 Connective cases

In this subsection we consider the quasi-category of comodules over $A \otimes A$ for a connective \mathbb{S} -algebra A . We show that the quasi-category of connective A -local spectra is equivalent to the quasi-category of connective comodules over $A \otimes A$ under some conditions.

We say that a spectrum X is connective if $\pi_i X = 0$ for all $i < 0$. We denote by $\mathrm{Sp}^{\geq 0}$ the full subcategory of Sp consisting of connective objects. In this subsection we let A be a connective \mathbb{S} -algebra and assume that $A_*(A)$ is flat as a left and right A_* -module.

We consider the condition that the multiplication map induces an isomorphism $\pi_0(A \otimes A) \xrightarrow{\cong} \pi_0 A$. Note that there is an isomorphism $\pi_0(A \otimes A) \cong \pi_0 A \otimes \pi_0 A$.

Let R be a (possibly non-commutative) ring with identity 1. The core cR of R is defined to be the subring

$$cR = \{r \in R \mid r \otimes 1 = 1 \otimes r \text{ in } R \otimes R\}$$

(see [3, 6.4]). The core cR is a commutative ring and the multiplication map $cR \otimes cR \rightarrow cR$ is an isomorphism. We see that, if the multiplication map induces an isomorphism $\pi_0(A \otimes A) \xrightarrow{\cong} \pi_0 A$, then $c\pi_0 A = \pi_0 A$ and, in particular, $\pi_0 A$ is a commutative ring.

Lemma 10. *Let M be a connective left $\Gamma(A)$ -comodule in Sp . If the multiplication map $A \otimes A \rightarrow A$ induces an isomorphism $\pi_0(A \otimes A) \xrightarrow{\cong} \pi_0 A$, then $P(M)$ is connective.*

Proof. We have the Bousfield-Kan spectral sequence

$$E_2^{s,t} \cong \pi^s \pi_t C^\bullet(A, A \otimes A, M) \implies \pi_{t-s} P(M).$$

This is an upper half plane spectral sequence. Since $A_*(A)$ is flat as a left and right A_* -module, there is an isomorphism

$$\pi_* C^r(A, A \otimes A, M) \cong C^r(A_*, A_*(A), M_*)$$

for any $r \geq 0$. Hence we obtain an isomorphism

$$E_2^{s,*} \cong H^s(\overline{C}^*(A_*, A_*(A), M_*)).$$

Let $\overline{\Gamma}$ be the kernel of the map $A_*(A) \rightarrow A_*$ induced by the multiplication. By the assumption that $\pi_0(A \otimes A) \xrightarrow{\cong} \pi_0(A)$, $\overline{\Gamma}_t = 0$ for $t \leq 0$. This implies that

$$\overline{C}^s(A_*, A_*(A), M_*)_t = 0$$

for $t < s$. Hence $E_2^{s,t} = 0$ for $t < s$. The lemma follows from [4, Lemma X.7.3]. \square

We recall that $L_A \mathbf{Sp}$ is the Bousfield localization of \mathbf{Sp} with respect to A and $L_A : \mathbf{Sp} \rightarrow \mathbf{Sp}$ is the associated localization functor and that we have the adjoint pair of functors

$$A \otimes (-) : L_A \mathbf{Sp} \rightleftarrows \mathbf{LComod}_{\Gamma(A)}(\mathbf{Sp}) : P.$$

We denote by $L_A \mathbf{Sp}^{\geq 0}$ the full subcategory of $L_A \mathbf{Sp}$ consisting of connective objects. We also denote by $\mathbf{LComod}_{\Gamma(A)}(\mathbf{Sp})^{\geq 0}$ the full subcategories of $\mathbf{LComod}_{\Gamma(A)}(\mathbf{Sp})$ consisting of connective objects.

By Lemma 10, we see that the functor $P : \mathbf{LComod}_{\Gamma(A)}(\mathbf{Sp}) \rightarrow L_A \mathbf{Sp}$ restricted to the full subcategory $\mathbf{LComod}_{\Gamma(A)}(\mathbf{Sp})^{\geq 0}$ factors through $L_A \mathbf{Sp}^{\geq 0}$ when the multiplication map induces an isomorphism $\pi_0(A \otimes A) \xrightarrow{\cong} \pi_0 A$. Hence we obtain the following corollary.

Corollary 4. *If the multiplication map induces an isomorphism $\pi_0(A \otimes A) \xrightarrow{\cong} \pi_0 A$, then there is an adjunction of functors*

$$A \otimes (-) : L_A \mathbf{Sp}^{\geq 0} \rightleftarrows \mathbf{LComod}_{\Gamma(A)}(\mathbf{Sp})^{\geq 0} : P.$$

We would like to show that the pair of functors $(A \otimes (-), P)$ is an adjoint equivalence under some conditions. In order to show that $A \otimes (-)$ is fully faithful, we have to show that the unit map $X \rightarrow P(A \otimes X)$ is an equivalence for any $X \in L_A \mathbf{Sp}^{\geq 0}$. We recall that the map $X \rightarrow P(A \otimes X)$ is an A -nilpotent completion in $\mathbf{Ho}(\mathbf{Sp})$ (see §5.2). The relation between the A -localization and the A -nilpotent completion was studied by Bousfield [3].

Lemma 11. *We assume that $\pi_0 A$ is isomorphic to $\mathbb{Z}/n\mathbb{Z}$ for some $n \geq 2$ or $\mathbb{Z}[J^{-1}]$ for some set J of primes. Then the functor $A \otimes (-) : L_A \mathbf{Sp}^{\geq 0} \rightarrow \mathbf{LComod}_{\Gamma(A)}(\mathbf{Sp})^{\geq 0}$ is fully faithful.*

Proof. Note that the multiplication map induces an isomorphism $\pi_0(A \otimes A) \xrightarrow{\cong} \pi_0(A)$ under the assumption and hence we have the adjunction of functors $A \otimes (-) : L_A \text{Sp}^{\geq 0} \rightleftarrows \text{LComod}_{\Gamma(A)}(\text{Sp})^{\geq 0} : P$ by Corollary 4.

We have to show that the unit map $X \rightarrow P(A \otimes X)$ of the adjunction $(A \otimes (-), P)$ is an equivalence for any $X \in L_A \text{Sp}^{\geq 0}$. By [3, Thm. 6.5 and 6.6], the A -localization $Y \rightarrow L_A Y$ is equivalent to the A -nilpotent completion $Y \rightarrow P(A \otimes Y)$ for any connective spectrum Y under the assumption. This implies that the map $X \rightarrow P(A \otimes X)$ is an equivalence for any $X \in L_A \text{Sp}^{\geq 0}$. \square

In order to show that the left adjoint $A \otimes (-)$ is essentially surjective, we need the following lemma.

Lemma 12. *We assume that the multiplication map induces an isomorphism $\pi_0(A \otimes A) \xrightarrow{\cong} \pi_0 A$. Let M be a connective left $\Gamma(A)$ -comodule. If $P(M) \simeq 0$, then $M \simeq 0$.*

Proof. We shall show that $P(M) \not\simeq 0$ if $M \not\simeq 0$. Suppose that $\pi_i M = 0$ for $i < n$ and $\pi_n M \neq 0$. Let $E_r^{s,t}$ be the Bousfield-Kan spectral sequence abutting to $\pi_* P(M)$. We have $E_2^{s,*} \cong H^s(\overline{C}^*(A_*, A_*(A), M_*))$. By the assumption, $E_2^{s,t} = 0$ for $t - s < n$ and $E_2^{0,n} \cong \pi_n M$. In particular, we have $E_\infty^{0,n} \neq 0$ and $\lim_r^1 E_r^{s,s+n} = 0$ for all $s \geq 0$. By [4, Lemma IX.5.4], $E_\infty^{0,n} \cong \text{Im}(\pi_n P(M) \rightarrow \pi_n M)$. Hence we obtain $\pi_n P(M) \neq 0$. \square

Theorem 10. *Let A be a connective \mathbb{S} -algebra such that $A_*(A)$ is flat as a left and right A_* -module. We assume that $\pi_0 A$ is isomorphic to \mathbb{Z}/n for some $n \geq 2$ or $\mathbb{Z}[J^{-1}]$ for some set J of primes. Then there is an adjoint equivalence of quasi-categories*

$$A \otimes (-) : L_A \text{Sp}^{\geq 0} \rightleftarrows \text{LComod}_{\Gamma(A)}(\text{Sp})^{\geq 0} : P.$$

Proof. By Corollary 4, we have the adjunction of functors $A \otimes (-) : L_A \text{Sp}^{\geq 0} \rightleftarrows \text{LComod}_{\Gamma(A)}(\text{Sp})^{\geq 0} : P$. By Lemma 11, the left adjoint $A \otimes (-)$ is fully faithful. Hence it suffices to show that $A \otimes (-)$ is essentially surjective.

Let $M \in \text{Comod}_{\Gamma(A)}(\text{Sp})^{\geq 0}$. By the counit of the adjunction, we have a map $\varepsilon : A \otimes P(M) \rightarrow M$. Let N be the cofiber of ε . Since the unit map $P(M) \rightarrow P(A \otimes P(M))$ is an equivalence, we see that $P(N) \simeq 0$. By Lemma 12, we obtain $N \simeq 0$ and hence $A \otimes P(M) \simeq M$. This shows that the left adjoint $A \otimes (-)$ is essentially surjective. \square

In the following of this subsection we shall consider some examples.

First, we consider the complex cobordism spectrum MU . The spectrum MU admits a commutative \mathbb{S} -algebra structure by [25]. Since $\pi_* MU = \mathbb{Z}[x_1, x_2, \dots]$ with $|x_i| = 2i$ for $i > 0$, MU is a connective commutative \mathbb{S} -algebra, the multiplication map induces an isomorphism $\pi_0(MU \otimes MU) \xrightarrow{\cong} \pi_0 MU$, and $c\pi_0 MU \cong \mathbb{Z}$. Note that the localization $L_{MU} X$ coincides with X for a connective spectrum X by [3, Thm. 3.1] and hence $L_{MU} \text{Sp}^{\geq 0}$ is equivalent to $\text{Sp}^{\geq 0}$. By Theorem 10, we obtain the following corollary.

Corollary 5 (cf. [10, 6.1.2]). *There is an adjoint equivalence*

$$MU \otimes (-) : \text{Sp}^{\geq 0} \rightleftarrows \text{LComod}_{\Gamma(MU)}(\text{Sp})^{\geq 0} : P.$$

Next, we consider the Brown-Peterson spectrum BP at a prime p . The spectrum BP admits an \mathbb{S} -algebra structure by [19, §2]. The coefficient ring of BP is a polynomial ring over the ring $\mathbb{Z}_{(p)}$ of integers localized at p with infinite many variables

$$\pi_*BP = \mathbb{Z}_{(p)}[v_1, v_2, \dots],$$

with degree $|v_i| = 2(p^i - 1)$ for $i \geq 1$. Hence BP is a connective \mathbb{S} -algebra, the multiplication map induces an isomorphism $\pi_0(BP \otimes BP) \xrightarrow{\cong} \pi_0BP$, and $c\pi_0BP \cong \mathbb{Z}_{(p)}$. Note that the localization $L_{BP}X$ coincides with the p -localization $X_{(p)}$ for a connective spectrum X by [3, Thm. 3.1] and hence $L_{BP}\mathrm{Sp}^{\geq 0}$ is equivalent to the full subcategory $\mathrm{Sp}_{(p)}^{\geq 0}$ of p -local spectra in $\mathrm{Sp}^{\geq 0}$. By Theorem 10, we obtain the following corollary.

Corollary 6. *There is an adjoint equivalence*

$$BP \otimes (-) : \mathrm{Sp}_{(p)}^{\geq 0} \rightleftarrows \mathrm{LComod}_{\Gamma(BP)}(\mathrm{Sp})^{\geq 0} : P.$$

Finally, we consider the mod p Eilenberg-Mac Lane spectrum $H\mathbb{F}_p$ for a prime p . We know that $H\mathbb{F}_p$ is a connective commutative \mathbb{S} -algebra. Since $\pi_0H\mathbb{F}_p \cong \mathbb{F}_p$, we see that the multiplication induces an isomorphism $\pi_0(H\mathbb{F}_p \otimes H\mathbb{F}_p) \xrightarrow{\cong} \pi_0H\mathbb{F}_p$ and $c\pi_0H\mathbb{F}_p \cong \mathbb{F}_p$. If X is a connective spectrum, then $L_{H\mathbb{F}_p}X$ is equivalent to the p -completion of X by [3, Thm. 3.1], and hence $L_{H\mathbb{F}_p}\mathrm{Sp}^{\geq 0}$ is equivalent to the full subcategory $(\mathrm{Sp}_p^\wedge)^{\geq 0}$ of p -complete spectra in $\mathrm{Sp}^{\geq 0}$. By Theorem 10, we obtain the following corollary.

Corollary 7 (cf. [10, 6.1.1]). *There is an adjoint equivalence*

$$H\mathbb{F}_p \otimes (-) : (\mathrm{Sp}_p^\wedge)^{\geq 0} \rightleftarrows \mathrm{LComod}_{\Gamma(H\mathbb{F}_p)}(\mathrm{Sp})^{\geq 0} : P.$$

5.6 A model of the $K(n)$ -local category

Let $K(n)$ be the n th Morava K -theory spectrum at a prime p and \mathbb{G}_n the n th Morava stabilizer group. In this subsection we show that the category of module objects over F_n in the $K(n)$ -local discrete symmetric \mathbb{G}_n -spectra models the $K(n)$ -local category, where F_n is a discrete model of the n th Morava E -theory spectrum.

The $K(n)$ -local category is the Bousfield localization of the stable homotopy category of spectra with respect to $K(n)$. It is known that the $K(n)$ -local categories for various n and p are fundamental building blocks of the stable homotopy category of spectra. Thus it is important to understand the $K(n)$ -local category.

Let E_n be the n th Morava E -theory spectrum at p . The Morava E -theory spectrum E_n is a commutative ring spectrum in the stable homotopy category of spectra and \mathbb{G}_n is identified with the group of multiplicative automorphisms of E_n . By Goerss-Hopkins [9], it was shown that the commutative ring spectrum structure on

E_n can be lifted to a unique E_∞ -ring spectrum structure up to homotopy. Furthermore, it was shown that \mathbb{G}_n acts on E_n in the category of E_∞ -ring spectra. There is a $K(n)$ -local E_n -based Adams spectral sequence abutting to the homotopy groups of the $K(n)$ -local sphere whose E_2 -page is the continuous cohomology groups of \mathbb{G}_n with coefficients in the homotopy groups of E_n . This suggests that the $K(n)$ -local sphere may be the \mathbb{G}_n -homotopy fixed points of E_n . Motivated by this observation, Devinatz-Hopkins [6] constructed a $K(n)$ -local E_∞ -ring spectrum E_n^{dhU} for any open subgroup U of \mathbb{G}_n , which has expected properties of the homotopy fixed points spectrum.

Davis [7] constructed a discrete \mathbb{G}_n -spectrum F_n which is defined by

$$F_n = \operatorname{colim}_U E_n^{dhU},$$

where U ranges over the open subgroups of \mathbb{G}_n . The spectrum F_n is a discrete model of E_n and actually we can recover E_n from F_n by the $K(n)$ -localization as

$$L_{K(n)}F_n \simeq E_n.$$

Furthermore, Behrens-Davis [5] upgraded the discrete \mathbb{G}_n -spectrum F_n to a commutative monoid object in the category $\Sigma\mathrm{Sp}(\mathbb{G}_n)$ of discrete symmetric \mathbb{G}_n -spectra, and showed that the unit map $L_{K(n)}\mathbb{S} \rightarrow F_n$ is a consistent $K(n)$ -local \mathbb{G}_n -Galois extension.

We can give a model structure on the category $\Sigma\mathrm{Sp}(\mathbb{G}_n)$ of discrete symmetric \mathbb{G}_n -spectra and consider the left Bousfield localization $\Sigma\mathrm{Sp}(\mathbb{G}_n)_{K(n)}$ with respect to $K(n)$ (see [5]). The category $\Sigma\mathrm{Sp}(\mathbb{G}_n)_{K(n)}$ is a left proper, combinatorial, $\Sigma\mathrm{Sp}$ -model category.

The unit map $\mathbb{S} \rightarrow F_n$ induces a symmetric monoidal $\Sigma\mathrm{Sp}$ -Quillen adjunction

$$\mathrm{Ex} : \Sigma\mathrm{Sp}_{K(n)} \rightleftarrows \mathrm{LMod}_{F_n}(\Sigma\mathrm{Sp}(\mathbb{G}_n)_{K(n)}) : \mathrm{Re},$$

where $\Sigma\mathrm{Sp}_{K(n)}$ is the left Bousfield localization of the category $\Sigma\mathrm{Sp}$ of symmetric spectra with respect to $K(n)$. In [31] we showed that the total left derived functor $\mathbb{L}\mathrm{Ex}$ of Ex is fully faithful as an $\mathrm{Ho}(\mathrm{Sp})$ -enriched functor. In this subsection we shall show that the adjunction is actually a Quillen equivalence and hence we can regard $\mathrm{LMod}_{F_n}(\Sigma\mathrm{Sp}(\mathbb{G}_n)_{K(n)})$ as a model of the $K(n)$ -local category.

We denote by $\mathrm{Sp}_{K(n)}$ the underlying quasi-category of the simplicial model category $\Sigma\mathrm{Sp}_{K(n)}$. The quasi-category $\mathrm{Sp}_{K(n)}$ is a stable homotopy theory with the tensor product $L_{K(n)}(- \otimes -)$ and the unit $L_{K(n)}\mathbb{S}$. Since we can regard E_n as an algebra object of $\mathrm{Sp}_{K(n)}$, we can consider the coalgebra $L_{K(n)}(E_n \otimes E_n)$ in ${}_{E_n}\mathrm{BMod}_{E_n}(\mathrm{Sp}_{K(n)})$ and the quasi-category of left $\Gamma(E_n)$ -comodules

$$\mathrm{LComod}_{\Gamma(E_n)}(\mathrm{Sp}_{K(n)}),$$

where $\Gamma(E_n) = (E_n, L_{K(n)}(E_n \otimes E_n))$.

Proposition 13. *We have an equivalence of quasi-categories*

$$L_{K(n)}(E_n \otimes (-)) : \mathrm{Sp}_{K(n)} \xrightarrow{\simeq} \mathrm{LComod}_{\Gamma(E_n)}(\mathrm{Sp}_{K(n)}).$$

Proof. We shall apply Theorem 7 for the stable homotopy theory $\mathrm{Sp}_{K(n)}$ and the algebra object E_n . The unit object $L_{K(n)}\mathbb{S}$ is E_n -nilpotent in $\mathrm{Sp}_{K(n)}$ by [6, Prop. A.3]. Note that a map $f : X \rightarrow Y$ in $\mathrm{Sp}_{K(n)}$ is an equivalence if and only if $L_{K(n)}(E_n \otimes f)$ is an equivalence since $K(n) \otimes E_n$ is a wedge of copies of $K(n)$. Hence $L_{E_n}(\mathrm{Sp}_{K(n)}) \simeq \mathrm{Sp}_{K(n)}$ and the proposition follows from Theorem 7. \square

We have an adjunction of quasi-categories

$$\mathrm{Sp}_{K(n)} \rightleftarrows \mathrm{LMod}_{E_n}(\mathrm{Sp}_{K(n)}), \quad (5)$$

where the left adjoint is given by smashing with E_n in $\mathrm{Sp}_{K(n)}$ and the right adjoint is the forgetful functor. By Theorem 6, we have an equivalence of quasi-categories

$$\mathrm{LComod}_{\Gamma(E_n)}(\mathrm{Sp}_{K(n)}) \simeq \mathrm{LComod}_{\Theta}(\mathrm{LMod}_{E_n}(\mathrm{Sp}_{K(n)})^{\mathrm{op}})^{\mathrm{op}},$$

where Θ is the comonad associated to adjunction (5). Hence we see that the forgetful functor $\mathrm{LMod}_{E_n}(\mathrm{Sp}_{K(n)}) \rightarrow \mathrm{Sp}_{K(n)}$ exhibits the quasi-category $\mathrm{Sp}_{K(n)}$ as comonadic over $\mathrm{LMod}_{E_n}(\mathrm{Sp}_{K(n)})$ (see also [23, Prop. 10.10]).

Let $\mathrm{Sp}(\mathbb{G}_n)_{K(n)}$ be the underlying quasi-categories of $\Sigma\mathrm{Sp}(\mathbb{G}_n)_{K(n)}$. We can regard F_n as an algebra object of $\mathrm{Sp}(\mathbb{G}_n)_{K(n)}$, and form a quasi-category

$$\mathrm{LMod}_{F_n}(\mathrm{Sp}(\mathbb{G}_n)_{K(n)})$$

of left modules objects over F_n in $\mathrm{Sp}(\mathbb{G}_n)_{K(n)}$. Note that $\mathrm{LMod}_{F_n}(\mathrm{Sp}(\mathbb{G}_n)_{K(n)})$ is equivalent to the underlying quasi-category of the symmetric monoidal $\Sigma\mathrm{Sp}$ -model category $\mathrm{LMod}_{F_n}(\Sigma\mathrm{Sp}(\mathbb{G}_n)_{K(n)})$. The adjunction $(\mathrm{Ex}, \mathrm{Re})$ of the $\Sigma\mathrm{Sp}$ -model categories induces an adjunction of the underlying quasi-categories

$$\mathcal{E}x : \mathrm{Sp}_{K(n)} \rightleftarrows \mathrm{LMod}_{F_n}(\mathrm{Sp}(\mathbb{G}_n)_{K(n)}) : \mathcal{R}e.$$

Let $U : \Sigma\mathrm{Sp}(\mathbb{G}_n)_{K(n)} \rightarrow \Sigma\mathrm{Sp}_{K(n)}$ be the forgetful functor. We can regard UF_n as a commutative monoid object in $\Sigma\mathrm{Sp}_{K(n)}$, and the unit map $\mathbb{S} \rightarrow UF_n$ induces a symmetric monoidal $\Sigma\mathrm{Sp}$ -Quillen adjunction

$$\Sigma\mathrm{Sp}_{K(n)} \rightleftarrows \mathrm{LMod}_{UF_n}(\Sigma\mathrm{Sp}_{K(n)}),$$

where the left adjoint is given by smashing with UF_n and the right adjoint is given by the forgetful functor. This induces an adjunction of the underlying quasi-categories

$$\mathrm{Sp}_{K(n)} \rightleftarrows \mathrm{LMod}_{UF_n}(\mathrm{Sp}_{K(n)}). \quad (6)$$

Hence we can consider the quasi-category of comodules

$$\mathrm{Comod}_{(UF_n, \Theta)}(\mathrm{Sp}_{K(n)})$$

associated to the adjunction and a map of quasi-categories

$$\mathrm{Coex} : \mathrm{Sp}_{K(n)} \longrightarrow \mathrm{Comod}_{(UF_n, \Theta)}(\mathrm{Sp}_{K(n)}).$$

In [31] we showed that there is an equivalence of quasi-categories

$$\mathrm{LMod}_{F_n}(\mathrm{Sp}(\mathbb{G}_n)_{K(n)}) \simeq \mathrm{Comod}_{(UF_n, \Theta)}(\mathrm{Sp}_{K(n)}),$$

and there is an equivalence of functors

$$\mathcal{E}x \simeq \mathrm{Coex}$$

under this equivalence.

Since the canonical map $UF_n \rightarrow E_n$ of commutative algebras is a weak equivalence in $\Sigma\mathrm{Sp}_{K(n)}$, we have a Quillen equivalence

$$\mathrm{LMod}_{UF_n}(\Sigma\mathrm{Sp}_{K(n)}) \xrightarrow{\simeq} \mathrm{LMod}_{E_n}(\Sigma\mathrm{Sp}_{K(n)}),$$

and hence we obtain an equivalence of the underlying quasi-categories

$$\mathrm{LMod}_{UF_n}(\mathrm{Sp}_{K(n)}) \xrightarrow{\simeq} \mathrm{LMod}_{E_n}(\mathrm{Sp}_{K(n)}).$$

Under this equivalence, we can identify two adjunctions (5) and (6), and hence the forgetful functor $\mathrm{LMod}_{UF_n}(\mathrm{Sp}_{K(n)}) \rightarrow \mathrm{Sp}_{K(n)}$ exhibits $\mathrm{Sp}_{K(n)}$ as comonadic over $\mathrm{LMod}_{UF_n}(\mathrm{Sp}_{K(n)})$, that is, the functor Coex is an equivalence of quasi-categories.

The adjunction $(\mathcal{E}x, \mathcal{R}e)$ of quasi-categories induces an adjunction of the homotopy categories

$$\mathrm{Ho}(\mathrm{Sp}_{K(n)}) \rightleftarrows \mathrm{Ho}(\mathrm{LMod}_{F_n}(\mathrm{Sp}(\mathbb{G}_n)_{K(n)})),$$

which is identified with the derived adjunction of the Quillen adjunction $(\mathrm{Ex}, \mathrm{Re})$. Since the functor Coex is equivalent to $\mathcal{E}x$ and is an equivalence of quasi-categories, the total left derived functor $\mathbb{L}\mathrm{Ex}$ is an equivalence of categories. Hence we obtain the following theorem.

Theorem 11. *The adjunction*

$$\mathrm{Ex} : \Sigma\mathrm{Sp}_{K(n)} \rightleftarrows \mathrm{Mod}_{F_n}(\Sigma\mathrm{Sp}(\mathbb{G}_n)_{K(n)}) : \mathrm{Re}$$

is a Quillen equivalence and hence the category $\mathrm{Mod}_{F_n}(\Sigma\mathrm{Sp}(\mathbb{G}_n)_{K(n)})$ models the $K(n)$ -local category.

6 Proof of Proposition 1

In this section we prove Proposition 1 stated in §3.1, which is technical but important for constructing a canonical map between opposite coCartesian fibrations. First, we give some basic examples of inner anodyne maps and study opposite marked anodyne maps. In §6.3 we introduce a marked simplicial set $\tilde{\mathcal{O}}(\Delta^n)^+$ in which the

underlying simplicial set is $\tilde{\mathcal{O}}(\Delta^n)$ and study inclusions of subcomplexes of the marked simplicial sets $\tilde{\mathcal{O}}(\Delta^n)^+$ and $(\tilde{\mathcal{O}}(\Delta^n)^+ \times (\Delta^{\{0\}})^b) \cup (\tilde{\mathcal{O}}(\Delta^n)^b \times (\Delta^1)^b)$. In §6.4 we give a proof of Proposition 1.

6.1 Examples of inner anodyne maps

In this subsection we give some basic examples of inner anodyne maps.

A map of simplicial sets is said to be inner anodyne if it has the left lifting property with respect to all inner fibrations. The class of inner anodyne maps is the smallest weakly saturated class of morphisms generated by all horn inclusions $\Lambda_i^n \hookrightarrow \Delta^n$ for $0 < i < n$.

For a sequence i_1, \dots, i_k of integers such that $0 \leq i_1 < \dots < i_k \leq n$, we denote by $\Lambda^n(i_1, \dots, i_k)$ the subcomplex $\bigcup_{i \neq i_1, \dots, i_k} d_i \Delta^{n-1}$ of Δ^n .

Lemma 13. *The inclusion $\Lambda^n(i_1, \dots, i_k) \hookrightarrow \Delta^n$ is an inner anodyne map for $k > 0$ and $0 < i_1 < \dots < i_k < n$.*

Proof. We shall prove the lemma by induction on k . When $k = 1$, the inclusion is the map $\Lambda_i^n \hookrightarrow \Delta^n$ for $0 < i = i_1 < n$ and hence it is an inner anodyne map. Suppose the lemma holds for $k - 1$ and we shall prove the lemma for k . The subcomplex $\Lambda^n(i_1, \dots, i_k) \cap d_{i_k} \Delta^{n-1}$ of $d_{i_k} \Delta^{n-1}$ is isomorphic to the subcomplex $\Lambda^{n-1}(i_1, \dots, i_{k-1})$ of Δ^{n-1} . By the hypothesis of induction, the inclusion $\Lambda^{n-1}(i_1, \dots, i_{k-1}) \hookrightarrow \Delta^{n-1}$ is an inner anodyne map. By the cobase change of the inclusion $\Lambda^{n-1}(i_1, \dots, i_{k-1}) \hookrightarrow \Delta^{n-1}$ along the map $\Lambda^{n-1}(i_1, \dots, i_{k-1}) \cong \Lambda^n(i_1, \dots, i_k) \cap d_{i_k} \Delta^{n-1} \hookrightarrow \Lambda^n(i_1, \dots, i_k)$, we see that the inclusion $\Lambda^n(i_1, \dots, i_k) \hookrightarrow \Lambda^n(i_1, \dots, i_{k-1})$ is an inner anodyne map. By the hypothesis of induction, the inclusion $\Lambda^n(i_1, \dots, i_{k-1}) \hookrightarrow \Delta^n$ is an inner anodyne map. Hence the composition $\Lambda^n(i_1, \dots, i_k) \hookrightarrow \Lambda^n(i_1, \dots, i_{k-1}) \hookrightarrow \Delta^n$ is also an inner anodyne map. \square

Lemma 14. *The inclusion $\Lambda^n(0, i_1, \dots, i_k) \hookrightarrow \Delta^n$ is an inner anodyne map for $k > 0$ and $1 < i_1 < \dots < i_k < n$.*

Proof. The subcomplex $\Lambda^n(0, i_1, \dots, i_k) \cap d_0 \Delta^{n-1}$ of $d_0 \Delta^{n-1}$ is isomorphic to the subcomplex $\Lambda^{n-1}(i_1 - 1, \dots, i_k - 1)$ of Δ^{n-1} . Since $0 < i_1 - 1 < \dots < i_k - 1 < n - 1$, the inclusion $\Lambda^{n-1}(i_1 - 1, \dots, i_k - 1) \hookrightarrow \Delta^{n-1}$ is an inner anodyne map by Lemma 13. By the cobase change of $\Lambda^{n-1}(i_1 - 1, \dots, i_k - 1) \hookrightarrow \Delta^{n-1}$ along the map $\Lambda^{n-1}(i_1 - 1, \dots, i_k - 1) \cong \Lambda^n(0, i_1, \dots, i_k) \cap d_0 \Delta^{n-1} \hookrightarrow \Lambda^n(0, i_1, \dots, i_k)$, we see that the inclusion $\Lambda^n(0, i_1, \dots, i_k) \hookrightarrow \Lambda^n(i_1, \dots, i_k)$ is an inner anodyne map. Since the inclusion $\Lambda^n(i_1, \dots, i_k) \hookrightarrow \Delta^n$ is an inner anodyne map by Lemma 13, the composition $\Lambda^n(0, i_1, \dots, i_k) \hookrightarrow \Lambda^n(i_1, \dots, i_k) \hookrightarrow \Delta^n$ is also an inner anodyne map. \square

6.2 Opposite marked anodyne maps

In this subsection we study opposite marked anodyne maps.

A marked simplicial set is a pair (K, \mathcal{E}) , where K is a simplicial set and \mathcal{E} is a set of edges of K that contains all degenerate edges. A map of marked simplicial sets $(K, \mathcal{E}) \rightarrow (L, \mathcal{E}')$ is a map of simplicial set $f : K \rightarrow L$ such that $f(\mathcal{E}) \subset \mathcal{E}'$. We denote by \mathbf{sSet}^+ the category of marked simplicial sets.

For a simplicial set K , we denote by K^\flat the marked simplicial set $(K, s_0(K_0))$, where $s_0(K_0)$ is the set of all degenerate edges of K , and by K^\sharp the marked simplicial set (K, K_1) , where K_1 is the set of all edges of K .

For a marked simplicial set (K, \mathcal{E}) , we have the opposite marked simplicial set $(K, \mathcal{E})^{\text{op}} = (K^{\text{op}}, \mathcal{E}^{\text{op}})$, where K^{op} is the opposite simplicial set of K and \mathcal{E}^{op} is the corresponding set of edges of K^{op} .

We say that a map of marked simplicial sets $K \rightarrow L$ is an opposite marked anodyne map if the opposite $K^{\text{op}} \rightarrow L^{\text{op}}$ is a marked anodyne map defined in [20, Def. 3.1.1.1]. The class of opposite marked anodyne maps in \mathbf{sSet}^+ is the smallest weakly saturated class of morphisms with the following properties:

1. For each $0 < i < n$, the inclusion $(\Lambda_i^n)^\flat \hookrightarrow (\Delta^n)^\flat$ is opposite marked anodyne.
2. For every $n > 0$, the inclusion

$$(\Lambda_0^n, (\Lambda_0^n)_1 \cap \mathcal{E}) \hookrightarrow (\Delta^n, \mathcal{E})$$

is opposite marked anodyne, where \mathcal{E} denotes the set of all degenerate edges of Δ^n together with the initial edge $\Delta^{\{0,1\}}$.

3. The inclusion

$$(\Lambda_1^2)^\sharp \coprod_{(\Lambda_1^2)^\flat} (\Delta^2)^\flat \hookrightarrow (\Delta^2)^\sharp$$

is opposite marked anodyne.

4. For every Kan complex K , the map $K^\flat \rightarrow K^\sharp$ is opposite marked anodyne.

Lemma 15. *The inclusion $(\Lambda_0^n)^\sharp \hookrightarrow (\Delta^n)^\sharp$ is opposite marked anodyne for $n > 0$.*

Proof. When $n = 1$, the lemma holds by property 2 of the class of opposite marked anodyne maps. We consider the case $n = 2$. By [20, Cor. 3.1.1.7], the inclusion $(\Lambda_0^2)^\sharp \coprod_{(\Lambda_0^2)^\flat} (\Delta^2)^\flat \hookrightarrow (\Delta^2)^\sharp$ is opposite marked anodyne. The inclusion $(\Lambda_0^2, (\Lambda_0^2)_1 \cap \mathcal{E}) \hookrightarrow (\Delta^2, \mathcal{E})$ is opposite marked anodyne by property 2 of the class of opposite marked anodyne maps, where \mathcal{E} is the set of edges of Δ^2 consisting of all degenerate edges together with $\Delta^{\{0,1\}}$. Taking the pushout of $(\Lambda_0^2, (\Lambda_0^2)_1 \cap \mathcal{E}) \hookrightarrow (\Delta^2, \mathcal{E})$ along the map $(\Lambda_0^2, (\Lambda_0^2)_1 \cap \mathcal{E}) \rightarrow (\Lambda_0^2)^\sharp$, we see that the inclusion $(\Lambda_0^2)^\sharp \hookrightarrow (\Lambda_0^2)^\sharp \coprod_{(\Lambda_0^2)^\flat} (\Delta^2)^\flat$ is opposite marked anodyne. Hence the composition $(\Lambda_0^2)^\sharp \hookrightarrow (\Lambda_0^2)^\sharp \coprod_{(\Lambda_0^2)^\flat} (\Delta^2)^\flat \hookrightarrow (\Delta^2)^\sharp$ is also opposite marked anodyne.

Now we consider the case $n \geq 3$. The inclusion $(\Lambda_0^n, (\Lambda_0^n)_1 \cap \mathcal{E}) \hookrightarrow (\Delta^n, \mathcal{E})$ is opposite marked anodyne by property 2 of the class of opposite marked anodyne maps, where \mathcal{E} is the set of edges of Δ^n consisting of all degenerate edges together with $\Delta^{\{0,1\}}$. Taking the pushout of $(\Lambda_0^n, (\Lambda_0^n)_1 \cap \mathcal{E}) \hookrightarrow (\Delta^n, \mathcal{E})$ along the map $(\Lambda_0^n, (\Lambda_0^n)_1 \cap \mathcal{E}) \rightarrow (\Lambda_0^n)^\sharp$, we see that the inclusion $(\Lambda_0^n)^\sharp \hookrightarrow (\Delta^n)^\sharp$ is opposite marked anodyne. \square

Lemma 16. *Let $K = (\Delta^n \times \partial\Delta^1) \cup (\Lambda_0^n \times \Delta^1)$ be the subcomplex of $\Delta^n \times \Delta^1$ for $n \geq 1$. Let \mathcal{E} be the set of edges of $\Delta^n \times \Delta^1$ consisting of all degenerate edges together with $\Delta^{\{0,1\}} \times \Delta^{\{0\}}$. The inclusion $(K, K_1 \cap \mathcal{E}) \hookrightarrow (\Delta^n \times \Delta^1, \mathcal{E})$ is an opposite marked anodyne map.*

Proof. Put $L(i) = (\Delta^{\{0,\dots,i\}} \times \Delta^{\{0\}}) \star (\Delta^{\{i,\dots,n\}} \times \Delta^{\{1\}})$ for $0 \leq i \leq n$. We set $\bar{L}(i) = K \cup (\cup_{j=0}^i L(j))$ for $0 \leq i \leq n$. Note that $\bar{L}(n) = \Delta^n \times \Delta^1$.

First, we show that the inclusion $K \hookrightarrow \bar{L}(n-1)$ is inner anodyne. Since $L(0) \cap K$ is isomorphic to Λ_1^{n+1} in $L(0) \cong \Delta^{n+1}$, we see that the inclusion $K \hookrightarrow \bar{L}(0)$ is inner anodyne. For $0 < i < n$, since $L(i) \cap \bar{L}(i-1)$ is isomorphic to $\Lambda^{n+1}(0, i+1)$ in $L(i) \cong \Delta^{n+1}$, we see that the inclusion $\bar{L}(i-1) \hookrightarrow \bar{L}(i)$ is inner anodyne by Lemma 14. Hence the composition $K \hookrightarrow \bar{L}(0) \hookrightarrow \dots \hookrightarrow \bar{L}(n-1)$ is also inner anodyne.

Since the class of inner anodyne maps is stable under the opposite, the inclusion $K^{\text{op}} \hookrightarrow \bar{L}(n-1)^{\text{op}}$ is also inner anodyne. By [20, Remark 3.1.1.4], we see that $K^{\flat} \hookrightarrow \bar{L}(n-1)^{\flat}$ is an opposite marked anodyne map. This implies that the inclusion $(K, K_1 \cap \mathcal{E}) \hookrightarrow (\bar{L}(n-1), \bar{L}(n-1)_1 \cap \mathcal{E})$ is opposite marked anodyne.

Now we consider the inclusion $\bar{L}(n-1) \hookrightarrow \bar{L}(n)$. We see that $L(n) \cap \bar{L}(n-1)$ is isomorphic to Λ_0^{n+1} in $L(n) \cong \Delta^{n+1}$. We can identify $(L(n), L(n)_1 \cap \mathcal{E})$ with $(\Delta^{n+1}, \mathcal{E}')$, where \mathcal{E}' is the set of edges of Δ^{n+1} consisting of all degenerate edges together with $\Delta^{\{0,1\}}$. Since the map $(\Lambda_0^{n+1}, (\Lambda_0^{n+1})_1 \cap \mathcal{E}') \hookrightarrow (\Delta^{n+1}, \mathcal{E}')$ is opposite marked anodyne, we see that $(\bar{L}(n-1), \bar{L}(n-1)_1 \cap \mathcal{E}) \hookrightarrow (\bar{L}(n), \mathcal{E})$ is opposite marked anodyne.

Therefore, the composition $(K, K_1 \cap \mathcal{E}) \hookrightarrow (\bar{L}(n-1), \bar{L}(n-1)_1 \cap \mathcal{E}) \hookrightarrow (\bar{L}(n), \mathcal{E})$ is also an opposite marked anodyne map. This completes the proof. \square

6.3 The marked simplicial set $\tilde{\mathcal{O}}(\Delta^n)^+$

In this subsection we introduce a marked simplicial set $\tilde{\mathcal{O}}(\Delta^n)^+$ in which the underlying simplicial set is $\tilde{\mathcal{O}}(\Delta^n)$. We study inclusions of subcomplexes of the marked simplicial sets $\tilde{\mathcal{O}}(\Delta^n)^+$ and $(\tilde{\mathcal{O}}(\Delta^n)^+ \times (\Delta^{\{0\}})^{\flat}) \cup (\tilde{\mathcal{O}}(\Delta^n)^{\flat} \times (\Delta^1)^{\flat})$.

Let $\tilde{\mathcal{E}}$ be the set of edges of $\tilde{\mathcal{O}}(\Delta^n)$ consisting of all non-degenerate edges together with edges $ij \rightarrow ik$ for $0 \leq i \leq j \leq k \leq n$. We regard the pair $(\tilde{\mathcal{O}}(\Delta^n), \tilde{\mathcal{E}})$ as a marked simplicial set. For a subcomplex K of $\tilde{\mathcal{O}}(\Delta^n)$, we set $\tilde{\mathcal{E}}_K = \tilde{\mathcal{E}} \cap K_1$ and denote by K^+ the marked simplicial set $(K, \tilde{\mathcal{E}}_K)$.

For $n > 0$, we let M_n be the subcomplex of $\tilde{\mathcal{O}}(\Delta^n)$ that contains all non-degenerate k -simplexes for $0 \leq k \leq n$ except for the n -simplex corresponding to $nn \rightarrow \dots \rightarrow 0n$.

Lemma 17. *The inclusion $\tilde{\mathcal{O}}(\partial\Delta^n)^+ \hookrightarrow M_n^+$ is an opposite marked anodyne map for all $n > 0$.*

Proof. First, we consider the case $n = 1$. We let B_0 be the 1-simplex corresponding to $00 \rightarrow 01$. The subcomplex $B_0 \cap \tilde{\mathcal{O}}(\partial\Delta^1)$ is isomorphic to Λ_0^1 in $B_0 \cong \Delta^1$. The

inclusion $(\Lambda_0^1)^\# \hookrightarrow (\Delta^1)^\#$ is opposite marked anodyne by Lemma 15. Taking the pushout of $(\Lambda_0^1)^\# \hookrightarrow (\Delta^1)^\#$ along the map $(\Lambda_0^1)^\# \cong B_0^+ \cap \tilde{\mathcal{O}}(\partial\Delta^1)^+ \rightarrow \tilde{\mathcal{O}}(\partial\Delta^1)^+$, we see that the inclusion $\tilde{\mathcal{O}}(\partial\Delta^1)^+ \hookrightarrow M_1^+$ is opposite marked anodyne.

Next, we consider the case $n = 2$. Let B_0 be the 2-simplex in $\tilde{\mathcal{O}}(\Delta^2)$ corresponding to $00 \rightarrow 01 \rightarrow 02$. The subcomplex $B_0^+ \cap \tilde{\mathcal{O}}(\partial\Delta^2)^+$ is isomorphic to $(\Lambda_0^2)^\#$ in $B_0^+ \cong (\Delta^2)^\#$. By Lemma 15, the inclusion $(\Lambda_0^2)^\# \hookrightarrow (\Delta^2)^\#$ is opposite marked anodyne. Taking the pushout of $(\Lambda_0^2)^\# \hookrightarrow (\Delta^2)^\#$ along the map $(\Lambda_0^2)^\# \cong B_0^+ \cap \tilde{\mathcal{O}}(\partial\Delta^2)^+ \rightarrow \tilde{\mathcal{O}}(\partial\Delta^2)^+$, we obtain an opposite marked anodyne map $\tilde{\mathcal{O}}(\partial\Delta^2)^+ \hookrightarrow \tilde{\mathcal{O}}(\partial\Delta^2)^+ \cup B_0^+$. Let $B_1(0)$ be the 2-simplex in $\tilde{\mathcal{O}}(\Delta^2)$ corresponding to $11 \rightarrow 01 \rightarrow 02$. The subcomplex $B_1(0) \cap (\tilde{\mathcal{O}}(\partial\Delta^2)^+ \cup B_0^+)$ is isomorphic to Λ_1^2 in Δ^2 . The inclusion $(\Lambda_1^2)^\flat \hookrightarrow (\Delta^2)^\flat$ is opposite marked anodyne. Taking the pushout of $(\Lambda_1^2)^\flat \hookrightarrow (\Delta^2)^\flat$ along the map $(\Lambda_1^2)^\flat \rightarrow \tilde{\mathcal{O}}(\partial\Delta^2)^+ \cup B_0^+$, we obtain an opposite marked anodyne map $\tilde{\mathcal{O}}(\partial\Delta^2)^+ \cup B_0^+ \rightarrow \tilde{\mathcal{O}}(\partial\Delta^2)^+ \cup B_0^+ \cup B_1(0)^+$. Let $B_1(1)$ be the 2-simplex in $\tilde{\mathcal{O}}(\Delta^2)$ corresponding to $11 \rightarrow 12 \rightarrow 02$. The subcomplex $B_1(1) \cap (\tilde{\mathcal{O}}(\partial\Delta^2)^+ \cup B_0^+ \cup B_1(0)^+)$ is isomorphic to Λ_0^2 in Δ^2 . The inclusion $(\Lambda_0^2, (\Lambda_0^2)_1 \cap \mathcal{E}') \hookrightarrow (\Delta^2, \mathcal{E}')$ is opposite marked anodyne, where \mathcal{E}' is the set of edges of Δ^2 consisting of all degenerate edges together with $\Delta^{\{0,1\}}$. Taking the pushout of $(\Lambda_0^2, (\Lambda_0^2)_1 \cap \mathcal{E}') \hookrightarrow (\Delta^2, \mathcal{E}')$ along the map $(\Lambda_0^2, (\Lambda_0^2)_1 \cap \mathcal{E}') \cong B_1(1)^+ \cap (\tilde{\mathcal{O}}(\partial\Delta^2)^+ \cup B_0^+ \cup B_1(0)^+) \rightarrow \tilde{\mathcal{O}}(\partial\Delta^2)^+ \cup B_0^+ \cup B_1(0)^+$, we see that the inclusion $\tilde{\mathcal{O}}(\partial\Delta^2)^+ \cup B_0^+ \cup B_1(0)^+ \rightarrow \tilde{\mathcal{O}}(\partial\Delta^2)^+ \cup B_0^+ \cup B_1(0)^+ \cup B_1(1)^+$ is opposite marked anodyne. Hence the composition $\tilde{\mathcal{O}}(\partial\Delta^2)^+ \rightarrow \tilde{\mathcal{O}}(\partial\Delta^2)^+ \cup B_0^+ \rightarrow \dots \rightarrow \tilde{\mathcal{O}}(\partial\Delta^2)^+ \cup B_0^+ \cup B_1(0)^+ \cup B_1(1)^+ = M_2^+$ is also opposite marked anodyne.

Now we assume $n \geq 3$. In this case we note that all edges of $\tilde{\mathcal{O}}(\Delta^n)$ is included in $\tilde{\mathcal{O}}(\partial\Delta^n)$. Let B_0^+ be the n -simplex in $\tilde{\mathcal{O}}(\Delta^n)^+$ corresponding to $00 \rightarrow 01 \rightarrow \dots \rightarrow 0n$. The subcomplex $B_0^+ \cap \tilde{\mathcal{O}}(\partial\Delta^n)^+$ of B_0^+ is isomorphic to $(\Lambda_0^n)^\#$ in $B_0^+ \cong (\Delta^n)^\#$. By Lemma 15, the inclusion $(\Lambda_0^n)^\# \hookrightarrow (\Delta^n)^\#$ is opposite marked anodyne. Taking the pushout of $(\Lambda_0^n)^\# \hookrightarrow (\Delta^n)^\#$ along the map $(\Lambda_0^n)^\# \cong B_0^+ \cap \tilde{\mathcal{O}}(\partial\Delta^n)^+ \rightarrow \tilde{\mathcal{O}}(\partial\Delta^n)^+$, we see that the inclusion $\tilde{\mathcal{O}}(\partial\Delta^n)^+ \hookrightarrow \tilde{\mathcal{O}}(\partial\Delta^n)^+ \cup B_0^+$ is opposite marked anodyne.

For $0 \leq i < n$, we let $\mathcal{L}(i)$ be the set of all paths from ii to $0n$ in diagram (1). To a path $l \in \mathcal{L}(i)$, we assign a sequence of integers $J(l) = (j_i, j_{i-1}, \dots, j_1)$ with $i \leq j_i \leq j_{i-1} \leq \dots \leq j_1 \leq n$ such that l is depicted as

$$\begin{array}{c}
 0j_1 \rightarrow \dots \rightarrow 0n \\
 \uparrow \\
 \dots \rightarrow 1j_1 \\
 \dots \\
 (i-1)j_i \rightarrow \dots \\
 \uparrow \\
 ii \rightarrow \dots \rightarrow ij_i
 \end{array}$$

We give $\{J(l) | l \in \mathcal{L}(i)\}$ the lexicographic order, and write $l < l'$ if $J(l) < J(l')$. This gives rise to a total order on $\mathcal{L}(i)$. For example, the path $ii \rightarrow \dots \rightarrow 0i \rightarrow \dots \rightarrow 0n$

is the smallest and the path $ii \rightarrow \dots \rightarrow in \rightarrow \dots \rightarrow 0n$ is the largest. For $l \in \mathcal{L}(i)$, we denote by $B(l)$ the n -simplex in $\tilde{\mathcal{O}}(\Delta^n)$ corresponding to l . Note that $\mathcal{L}(0)$ consists of a unique element l_0 and that $B(l_0) = B_0$. We set $B_i = \bigcup_{l \in \mathcal{L}(i)} B(l)$ and $\bar{B}_i = \tilde{\mathcal{O}}(\partial\Delta^n) \cup \bigcup_{j=0}^i B_j$. We shall show that the inclusion $\bar{B}_{i-1}^+ \hookrightarrow \bar{B}_i^+$ is opposite marked anodyne for $0 < i < n$.

For $0 < i < n$ and $l \in \mathcal{L}(i)$, we set $\bar{B}(l) = \bar{B}_{i-1} \cup \bigcup_{l' \leq l} B(l')$ and $\bar{B}(l)^\circ = \bar{B}_{i-1} \cup \bigcup_{l' < l} B(l')$. It suffices to show that the inclusion $\bar{B}(l)^{\circ+} \hookrightarrow \bar{B}(l)^+$ is opposite marked anodyne for all $l \in \mathcal{L}(i)$.

Let l_i be the path $ii \rightarrow \dots \rightarrow 0i \rightarrow \dots \rightarrow 0n$ for $0 < i < n$. The subcomplex $B(l_i) \cap \bar{B}_{i-1}$ of $B(l_i)$ is isomorphic to Λ_i^n of Δ^n . The inclusion $(\Lambda_i^n)^b \hookrightarrow (\Delta^n)^b$ is opposite marked anodyne. Taking the pushout of $(\Lambda_i^n)^b \hookrightarrow (\Delta^n)^b$ along the map $(\Lambda_i^n)^b \cong B(l_i)^b \cap \bar{B}_{i-1}^+ \rightarrow \bar{B}_{i-1}^+$, we see that the inclusion $\bar{B}_{i-1}^+ \hookrightarrow \bar{B}_{i-1}^+ \cup B(l_i)^+$ is opposite marked anodyne.

Let l'_i be the path $ii \rightarrow \dots \rightarrow in \rightarrow \dots \rightarrow 0n$. We take $l \in \mathcal{L}(i)$ such that $l_i < l < l'_i$. Let $\{\alpha_1, \dots, \alpha_k\}$ ($0 < \alpha_1 < \dots < \alpha_k < n$) be the set of integers such that the sub-path $l(\alpha_t - 1) \rightarrow l(\alpha_t) \rightarrow l(\alpha_t + 1)$ of l is depicted as

$$\begin{array}{ccc} a, b & \rightarrow & a, b + 1 \\ \uparrow & & \\ a + 1, b & & \end{array}$$

for $t = 1, \dots, k$. We consider the subcomplex $B(l) \cap \bar{B}(l)^\circ$ of $B(l)$. There are two cases. (1) If the first edge of l is $ii \rightarrow (i-1)i$, then the subcomplex $B(l) \cap \bar{B}(l)^\circ$ of $B(l)$ is isomorphic to the subcomplex $\Lambda^n(\alpha_1, \dots, \alpha_k)$ of Δ^n . Since the inclusion $\Lambda^n(\alpha_1, \dots, \alpha_k) \hookrightarrow \Delta^n$ is inner anodyne by Lemma 13, the inclusion $(\Lambda^n(\alpha_1, \dots, \alpha_k))^b \hookrightarrow (\Delta^n)^b$ is opposite marked anodyne. Taking the pushout of $(\Lambda^n(\alpha_1, \dots, \alpha_k))^b \hookrightarrow (\Delta^n)^b$ along the map $(\Lambda^n(\alpha_1, \dots, \alpha_k))^b \cong B(l)^b \cap \bar{B}(l)^{\circ+} \rightarrow \bar{B}(l)^{\circ+}$, we see that the inclusion $\bar{B}(l)^{\circ+} \hookrightarrow \bar{B}(l)^+$ is opposite marked anodyne. (2) If the first edge of l is $ii \rightarrow i(i+1)$, then the subcomplex $B(l) \cap \bar{B}(l)^\circ$ of $B(l)$ is isomorphic to the subcomplex $\Lambda^n(0, \alpha_1, \dots, \alpha_k)$ of Δ^n . Note that $\alpha_1 > 1$ in this case. Since the inclusion $\Lambda^n(0, \alpha_1, \dots, \alpha_k) \hookrightarrow \Delta^n$ is inner anodyne by Lemma 14, the inclusion $(\Lambda^n(0, \alpha_1, \dots, \alpha_k))^b \hookrightarrow (\Delta^n)^b$ is opposite marked anodyne. Taking the pushout of $(\Lambda^n(0, \alpha_1, \dots, \alpha_k))^b \hookrightarrow (\Delta^n)^b$ along the map $(\Lambda^n(0, \alpha_1, \dots, \alpha_k))^b \cong B(l)^b \cap \bar{B}(l)^{\circ+} \rightarrow \bar{B}(l)^{\circ+}$, we see that the inclusion $\bar{B}(l)^{\circ+} \hookrightarrow \bar{B}(l)^+$ is opposite marked anodyne.

Finally, we shall show that $\bar{B}(l'_i)^{\circ+} \hookrightarrow \bar{B}(l'_i)^+$ is opposite marked anodyne for $0 < i < n$. The subcomplex $B(l'_i) \cap \bar{B}(l'_i)^\circ$ is isomorphic to the subcomplex Λ_0^n of Δ^n . Note that $ii \rightarrow i(i+1)$ is a marked edge, which corresponds to $\Delta^{\{0,1\}}$ under the isomorphism $\Lambda_0^n \cong B(l'_i) \cap \bar{B}(l'_i)^\circ$. The inclusion $(\Lambda_0^n, (\Lambda_0^n) \cap \mathcal{E}') \hookrightarrow (\Delta^n, \mathcal{E}')$ is opposite marked anodyne, where \mathcal{E}' is the set of edges of Δ^n consisting of all degenerate edges together with $\Delta^{\{0,1\}}$. Taking the pushout of $(\Lambda_0^n, (\Lambda_0^n) \cap \mathcal{E}') \hookrightarrow (\Delta^n, \mathcal{E}')$ along the map $(\Lambda_0^n, (\Lambda_0^n) \cap \mathcal{E}') \rightarrow B(l'_i)^+ \cap \bar{B}(l'_i)^{\circ+} \rightarrow \bar{B}(l'_i)^{\circ+}$, we see that the inclusion $\bar{B}(l'_i)^{\circ+} \hookrightarrow \bar{B}(l'_i)^+$ is opposite marked anodyne. This completes the proof. \square

For $n > 0$, we let

$$\begin{aligned}\tilde{A} &= (\tilde{\mathcal{O}}(\Delta^n) \times \Delta^{\{0\}}) \cup (\tilde{\mathcal{O}}(\partial\Delta^n) \times \Delta^1), \\ \tilde{B} &= \tilde{A} \cup (M_n \times \Delta^{\{1\}}), \\ \tilde{C} &= \tilde{A} \cup (M_n \times \Delta^1)\end{aligned}$$

be the subcomplexes of $\tilde{\mathcal{O}}(\Delta^n) \times \Delta^1$. We denote by $(\tilde{\mathcal{O}}(\Delta^n) \times \Delta^1)^+$ the marked simplicial set $(\tilde{\mathcal{O}}(\Delta^n)^+ \times (\Delta^{\{0\}})^b) \cup (\tilde{\mathcal{O}}(\Delta^n)^b \times (\Delta^1)^b)$. For a subcomplex K of $\tilde{\mathcal{O}}(\Delta^n) \times \Delta^1$, we denote by K^+ the subcomplex of the marked simplicial set $(\tilde{\mathcal{O}}(\Delta^n) \times \Delta^1)^+$ in which the underlying simplicial set is K .

Lemma 18. *The inclusion $\tilde{B}^+ \hookrightarrow \tilde{C}^+$ is an opposite marked anodyne map.*

Proof. We use the notation in the proof of Lemma 17. Recall that B_0 is the n -simplex in $\tilde{\mathcal{O}}(\Delta^n)$ corresponding to $00 \rightarrow 01 \rightarrow \dots \rightarrow 0n$. Since the subcomplex $B_0 \cap \tilde{\mathcal{O}}(\partial\Delta^n)$ of B_0 is isomorphic to Λ_0^n in Δ^n , the subcomplex $\tilde{B} \cap (B_0 \times \Delta^1)$ of $B_0 \times \Delta^1$ is isomorphic to the subcomplex $(\Delta^n \times \partial\Delta^1) \cup (\Lambda_0^n \times \Delta^1)$ of $\Delta^n \times \Delta^1$. Since $00 \rightarrow 01$ is a marked edge of $\tilde{\mathcal{O}}(\Delta^n)^+$, we see that the inclusion $\tilde{B}^+ \hookrightarrow \tilde{B}^+ \cup (B_0 \times \Delta^1)^+$ is opposite marked anodyne by using Lemma 16.

We set $C_i = \tilde{B} \cup (\bar{B}_i \times \Delta^1)$. We shall show that the inclusion $C_{i-1}^+ \hookrightarrow C_i^+$ is opposite marked anodyne for $0 < i < n$. For this purpose, it suffices to show that the inclusion $C_{i-1}^+ \cup (\bar{B}(l)^\circ \times \Delta^1)^+ \hookrightarrow C_{i-1}^+ \cup (\bar{B}(l) \times \Delta^1)^+$ is opposite marked anodyne for all $l \in \mathcal{L}(i)$.

Recall that l_i is the path $ii \rightarrow \dots \rightarrow 0i \rightarrow \dots \rightarrow 0n$ and that the subcomplex $B(l_i) \cap \bar{B}_{i-1}$ of $B(l_i)$ is isomorphic to Λ_i^n of Δ^n . This implies that $(B(l_i) \times \Delta^1) \cap C_{i-1}$ is isomorphic to $(\Lambda_i^n \times \Delta^1) \cup (\Delta^n \times \partial\Delta^1)$. The inclusion $(\Lambda_i^n \times \Delta^1) \cup (\Delta^n \times \partial\Delta^1) \hookrightarrow (\Delta^n \times \Delta^1)$ is inner anodyne for $0 < i < n$ by [20, Cor. 2.3.2.4]. This implies that $(\Lambda_i^n \times \Delta^1)^b \cup (\Delta^n \times \partial\Delta^1)^b \hookrightarrow (\Delta^n \times \Delta^1)^b$ is opposite marked anodyne. Hence we see that the inclusion $C_{i-1}^+ \hookrightarrow C_{i-1}^+ \cup (B(l_i) \times \Delta^1)^+$ is opposite marked anodyne.

We take $l \in \mathcal{L}(i)$ such that $l_i < l < l'_i$, where l'_i is the path $ii \rightarrow \dots \rightarrow in \rightarrow \dots \rightarrow 0n$. Let $\{\alpha_1, \dots, \alpha_k\}$ ($0 < \alpha_1 < \dots < \alpha_k < n$) be the set of integers such that the sub-path $l(\alpha - 1) \rightarrow l(\alpha) \rightarrow l(\alpha + 1)$ is $a + 1, b \rightarrow a, b \rightarrow a, b + 1$ for some a, b .

Recall that the subcomplex $B(l) \cap \bar{B}(l)^\circ$ of $B(l)$ is isomorphic to the subcomplex $\Lambda^n(\alpha_1, \dots, \alpha_k)$ of Δ^n , if the first edge of l is $ii \rightarrow (i-1)i$. This implies that $(B(l) \times \Delta^1) \cap (C_{i-1} \cup (\bar{B}(l)^\circ \times \Delta^1))$ is isomorphic to $(\Lambda^n(\alpha_1, \dots, \alpha_k) \times \Delta^1) \cup (\Delta^n \times \partial\Delta^1)$. In this case the inclusion $(\Lambda^n(\alpha_1, \dots, \alpha_k) \times \Delta^1) \cup (\Delta^n \times \partial\Delta^1) \hookrightarrow \Delta^n \times \Delta^1$ is inner anodyne by Lemma 13 and [20, Cor. 2.3.2.4]. This implies that $(\Lambda^n(\alpha_1, \dots, \alpha_k) \times \Delta^1)^b \cup (\Delta^n \times \partial\Delta^1)^b \hookrightarrow (\Delta^n \times \Delta^1)^b$ is opposite marked anodyne. Hence we see that $C_{i-1}^+ \cup (\bar{B}(l)^\circ \times \Delta^1)^+ \hookrightarrow C_{i-1}^+ \cup (\bar{B}(l) \times \Delta^1)^+$ is opposite marked anodyne in this case.

If the first edge of l is $ii \rightarrow i(i+1)$, then the subcomplex $B(l) \cap \bar{B}(l)^\circ$ of $B(l)$ is isomorphic to the subcomplex $\Lambda^n(0, \alpha_1, \dots, \alpha_k)$ of Δ^n , where $\alpha_1 > 1$. This implies that $(B(l) \times \Delta^1) \cap (C_{i-1} \cup (\bar{B}(l)^\circ \times \Delta^1))$ is isomorphic to $(\Lambda^n(0, \alpha_1, \dots, \alpha_k) \times \Delta^1) \cup (\Delta^n \times \partial\Delta^1)$. In this case the inclusion $(\Lambda^n(0, \alpha_1, \dots, \alpha_k) \times \Delta^1) \cup (\Delta^n \times \partial\Delta^1) \hookrightarrow \Delta^n \times \Delta^1$ is inner anodyne by Lemma 13 and [20, Cor. 2.3.2.4]. This implies that $(\Lambda^n(0, \alpha_1, \dots, \alpha_k) \times \Delta^1)^b \cup (\Delta^n \times \partial\Delta^1)^b \hookrightarrow (\Delta^n \times \Delta^1)^b$ is opposite marked anodyne. Hence we see that $C_{i-1}^+ \cup (\bar{B}(l)^\circ \times \Delta^1)^+ \hookrightarrow C_{i-1}^+ \cup (\bar{B}(l) \times \Delta^1)^+$ is opposite marked anodyne in this case.

$\Delta^n \times \Delta^1$ is inner anodyne by Lemma 14 and [20, Cor. 2.3.2.4]. This implies that $(\Delta^n(0, \alpha_1, \dots, \alpha_k) \times \Delta^1)^b \cup (\Delta^n \times \partial \Delta^1)^b \hookrightarrow (\Delta^n \times \Delta^1)^b$ is opposite marked anodyne. Hence we see that $C_{i-1}^+ \cup (\overline{B}(l)^\circ \times \Delta^1)^+ \hookrightarrow C_{i-1}^+ \cup (\overline{B}(l) \times \Delta^1)^+$ is also opposite marked anodyne in this case.

Finally, we shall show that $C_{i-1}^+ \cup (\overline{B}(l'_i)^\circ \times \Delta^1)^+ \hookrightarrow C_{i-1}^+ \cup (\overline{B}(l'_i) \times \Delta^1)^+$ is opposite marked anodyne. Since the subcomplex $B(l'_i) \cap \overline{B}(l'_i)^\circ$ is isomorphic to the subcomplex Λ_0^n of Δ^n , the subcomplex $(B(l'_i) \times \Delta^1) \cap (C_{i-1} \cup (\overline{B}(l'_i)^\circ \times \Delta^1))$ is isomorphic to $(\Lambda_0^n \times \Delta^1) \cup (\Delta^n \times \partial \Delta^1)$. Since $ii \rightarrow i(i+1)$ is a marked edge of $\tilde{\mathcal{C}}(\Delta^n)$, we see that $C_{i-1}^+ \cup (\overline{B}(l'_i)^\circ \times \Delta^1)^+ \hookrightarrow C_{i-1}^+ \cup (\overline{B}(l'_i) \times \Delta^1)^+$ is opposite marked anodyne by using Lemma 16. This completes the proof. \square

6.4 Proof of Proposition 1

In this subsection we give a proof of Proposition 1. For this purpose, we show that the map $\pi_X : \mathcal{R} \rightarrow RX$ has right lifting property with respect to the maps $\partial \Delta^n \hookrightarrow \Delta^n$ for $n > 0$ if the final vertex $\Delta^{\{n\}}$ goes to an object of \mathcal{R}^0 .

Let $p : X \rightarrow S$ and $q : Y \rightarrow S$ be coCartesian fibrations over a quasi-category S . Suppose we have a map $G : Y \rightarrow X$ over S such that G_s admits a left adjoint F_s for all $s \in S$.

We recall that

$$\mathcal{R} = RX \times_{H(\text{Fun}(\Delta^{\{0\}}, X))} H(\text{Fun}^S(\Delta^1, X)) \times_{H(\text{Fun}(\Delta^{\{1\}}, X))} RY.$$

We have the projection map $\pi_X : \mathcal{R} \rightarrow RX$.

We identify objects of RX with objects of X . For $x \in X$ with $s = p(x)$, we have an object $(x, u_x, F_s(x))$ of \mathcal{R} over s , where $u_x : x \rightarrow G_s F_s(x)$ is the unit map of the adjunction (F_s, G_s) at x .

The following is a key lemma.

Lemma 19. *Suppose we have a commutative diagram*

$$\begin{array}{ccc} \partial \Delta^n & \xrightarrow{f} & \mathcal{R} \\ \downarrow & \nearrow & \downarrow \pi_X \\ \Delta^n & \xrightarrow{g} & RX \end{array}$$

for $n > 0$, where the left vertical arrow is the inclusion. We put $x = g(\Delta^{\{n\}})$ and $s = p(x)$. If $f(\Delta^{\{n\}}) = (x, u_x, F_s(x))$, then there exists a dotted arrow $\Delta^n \rightarrow \mathcal{R}$ making the whole diagram commutative.

Proof (Proof of Proposition 1). By Lemma 19, the map $\pi_X^0 : \mathcal{R}^0 \rightarrow RX$ has the right lifting property with respect to the maps $\partial \Delta^n \hookrightarrow \Delta^n$ for all $n \geq 0$. Hence π_X^0 is a trivial Kan fibration. \square

In order to prove Lemma 19, we consider the following situation.

Let h be a map $\tilde{\mathcal{O}}(\Delta^n) \rightarrow S$ for $n > 0$ such that $h(ii) \rightarrow \cdots \rightarrow h(0i)$ is a totally degenerate simplex in S for all $0 \leq i \leq n$. We set $\bar{h} = h\pi$, where $\pi : \tilde{\mathcal{O}}(\Delta^n) \times \Delta^1 \rightarrow \tilde{\mathcal{O}}(\Delta^n)$ is the projection.

Let X^\natural be the marked simplicial set in which the simplicial set is X and the set of marked edges consists of all p -coCartesian edges. Suppose that we have an n -simplex in RX that is represented by $g : \tilde{\mathcal{O}}(\Delta^n) \rightarrow X$ covering h . Note that we can regard g as a map of marked simplicial sets $\tilde{\mathcal{O}}(\Delta^n)^+ \rightarrow X^\natural$.

Furthermore, we suppose that we have a map $\partial\Delta^n \rightarrow \mathcal{R}$ that is represented by a triple of maps (g', k, f) , where $g' : \tilde{\mathcal{O}}(\partial\Delta^n) \rightarrow X$, $k : \tilde{\mathcal{O}}(\partial\Delta^n) \rightarrow \text{Fun}^S(\Delta^1, X)$, and $f : \tilde{\mathcal{O}}(\partial\Delta^n) \rightarrow Y$. We assume that g' is the restriction of g . Then the maps g', k , and f cover h , respectively.

Let Y^\natural be the marked simplicial set defined in the same way as X^\natural . We can regard f as a map of marked simplicial sets $\tilde{\mathcal{O}}(\partial\Delta^n)^+ \rightarrow Y^\natural$. There is an extension \tilde{f} of f to M_n^+ covering h by Lemma 17.

We recall that \tilde{A} , \tilde{B} , and \tilde{C} are subcomplexes of $\tilde{\mathcal{O}}(\Delta^n) \times \Delta^1$ given by $\tilde{A} = (\tilde{\mathcal{O}}(\Delta^n) \times \Delta^{\{0\}}) \cup (\tilde{\mathcal{O}}(\partial\Delta^n) \times \Delta^1)$, $\tilde{B} = \tilde{A} \cup (M_n \times \Delta^{\{1\}})$, $\tilde{C} = \tilde{A} \cup (M_n \times \Delta^1)$. Using the maps g, k , and $G(\tilde{f})$, we obtain a map of marked simplicial sets $\tilde{B}^+ \rightarrow X^\natural$ over \bar{h} . Furthermore, by Lemma 18, we can extend this map to a map of marked simplicial sets $w : \tilde{C}^+ \rightarrow X^\natural$ covering \bar{h} .

Let D be the n -simplex of $\tilde{\mathcal{O}}(\Delta^n)$ corresponding to $nn \rightarrow \cdots \rightarrow 0n$. By restricting w to $(D \times \Delta^{\{0\}}) \cup (\partial D \times \Delta^1)$, we obtain a map $v : (D \times \Delta^{\{0\}}) \cup (\partial D \times \Delta^1) \rightarrow X_s$, where $s = h(nn)$. We denote by g_D the restriction of g to D and by $\tilde{f}_{\partial D}$ the restriction of \tilde{f} to ∂D . Note that the restriction of v to $D \times \Delta^{\{0\}}$ is identified with g_D and that the restriction of v to $\partial D \times \Delta^{\{1\}}$ is $G_s(\tilde{f}_{\partial D})$.

We would like to have maps $\tilde{f}_D : D \rightarrow Y_s$ and $\bar{v} : D \times \Delta^1 \rightarrow X_s$ such that \tilde{f}_D is an extension of $\tilde{f}_{\partial D}$, \bar{v} is an extension of v , and the restriction of \bar{v} to $D \times \Delta^{\{1\}}$ is $G(\tilde{f}_D)$. Hence, in order to prove Lemma 19, it suffices to prove the following lemma.

Lemma 20. *Let $L : \mathcal{C} \rightleftarrows \mathcal{D} : R$ be an adjunction of quasi-categories. Suppose we have maps $f : (\Delta^n \times \Delta^{\{0\}}) \cup (\partial\Delta^n \times \Delta^1) \rightarrow \mathcal{C}$ and $g : \partial\Delta^n \rightarrow \mathcal{D}$ for $n > 0$ such that $Rg = f|_{\partial\Delta^n \times \Delta^{\{1\}}}$. We put $c = f(\Delta^{\{0\}} \times \Delta^{\{0\}})$ and $d = g(\Delta^{\{0\}})$. If $g(d) = L(c)$ and $f(\Delta^{\{0\}} \times \Delta^1)$ is the unit map $c \rightarrow RL(c)$ of the adjunction (L, R) at c , then there exist maps $F : \Delta^n \times \Delta^1 \rightarrow \mathcal{C}$ and $G : \Delta^n \rightarrow \mathcal{D}$ such that F is an extension of f , G is an extension of g , and $RG = F|_{\Delta^n \times \Delta^{\{1\}}}$.*

Proof. Let $\pi : \mathcal{M} \rightarrow \Delta^1$ be a map associated to the adjunction (L, R) , which is a coCartesian fibration and a Cartesian fibration. We may assume that the fibers $\mathcal{M}_{\{0\}}$ and $\mathcal{M}_{\{1\}}$ over $\{0\}$ and $\{1\}$ are isomorphic to \mathcal{C} and \mathcal{D} , respectively. We regard f as a map $(\Delta^n \times \Delta^{\{0\}}) \cup (\partial\Delta^n \times \Delta^1) \rightarrow \mathcal{M}_{\{0\}}$ and g as a map $\partial\Delta^n \rightarrow \mathcal{M}_{\{1\}}$.

Since $\mathcal{M} \rightarrow \Delta^1$ is a Cartesian fibration, we can extend the map g to a map $h : \partial\Delta^n \times \Delta^1 \rightarrow \mathcal{M}$ such that $h|_{\partial\Delta^n \times \Delta^{\{0\}}} = Rg$, $h|_{\partial\Delta^n \times \Delta^{\{1\}}} = g$, and $h(\Delta^{\{i\}} \times \Delta^1)$ is a π -Cartesian edge over Δ^1 for all $i = 0, 1, \dots, n$. By the assumption that

$Rg = f|_{\partial\Delta^n \times \Delta^1\{1\}}$, we obtain a map $k : \partial\Delta^n \times \Lambda_1^2 \rightarrow \mathcal{M}$ such that $k|_{\partial\Delta^n \times \Delta^1\{0,1\}} = f|_{\partial\Delta^n \times \Delta^1\{0,1\}}$ and $k|_{\partial\Delta^n \times \Delta^1\{1,2\}} = h$.

By the assumptions that $g(d) = L(c)$ and $f(\Delta^{\{0\}} \times \Delta^1)$ is the unit map $c \rightarrow RL(c)$, we have a map $l : \Delta^{\{0\}} \times \Delta^2 \rightarrow \mathcal{M}$ such that $l|_{\Delta^{\{0\}} \times \Delta^1\{0,1\}} = f|_{\Delta^{\{0\}} \times \Delta^1}$, $l|_{\Delta^{\{0\}} \times \Delta^1\{1,2\}} = k|_{\Delta^{\{0\}} \times \Delta^1\{1,2\}}$, and $l(\Delta^{\{0\}} \times \Delta^{\{0,2\}})$ is π -coCartesian.

Hence we obtain a map $k \cup l : (\partial\Delta^n \times \Lambda_1^2) \cup (\Delta^{\{0\}} \times \Delta^2) \rightarrow \mathcal{M}$. Let $\sigma : \Delta^n \times \Delta^2 \rightarrow \Delta^1$ be the projection $\Delta^n \times \Delta^2 \rightarrow \Delta^2$ followed by $s^0 : \Delta^2 \rightarrow \Delta^1$, where $s^0(\{0\}) = s^0(\{1\}) = \{0\}$ and $s^0(\{2\}) = \{1\}$. We shall show that $k \cup l$ extends to a map on $\Delta^n \times \Delta^2$ covering σ .

Since $\Lambda_1^2 \hookrightarrow \Delta^2$ is inner anodyne, $(\partial\Delta^n \times \Lambda_1^2) \cup (\Delta^{\{0\}} \times \Delta^2) \rightarrow \partial\Delta^n \times \Delta^2$ is also inner anodyne by [20, Cor. 2.3.2.4]. Hence there is an extension $m : \partial\Delta^n \times \Delta^2 \rightarrow \mathcal{M}$ of $k \cup l : (\partial\Delta^n \times \Lambda_1^2) \cup (\Delta^{\{0\}} \times \Delta^2) \rightarrow \mathcal{M}$ covering σ .

We have the map $f'|_{\Delta^n \times \Delta^1\{0\}} \cup m|_{\partial\Delta^n \times \Delta^1\{0,2\}} : (\Delta^n \times \Delta^1\{0\}) \cup (\partial\Delta^n \times \Delta^1\{0,2\}) \rightarrow \mathcal{M}$. Since $m(\Delta^{\{0\}} \times \Delta^1\{0,2\})$ is a π -coCartesian edge over Δ^1 , there is an extension $p(0,2) : \Delta^n \times \Delta^1\{0,2\} \rightarrow \mathcal{M}$ of $f'|_{\Delta^n \times \Delta^1\{0\}} \cup m|_{\partial\Delta^n \times \Delta^1\{0,2\}}$ covering σ by [20, Prop. 2.4.1.8].

We have the map $p(0,2)|_{\Delta^n \times \Delta^1\{2\}} \cup m|_{\partial\Delta^n \times \Delta^1\{1,2\}} : (\Delta^n \times \Delta^1\{2\}) \cup (\partial\Delta^n \times \Delta^1\{1,2\}) \rightarrow \mathcal{M}$. Since $m(\Delta^{\{n\}} \times \Delta^1\{1,2\})$ is a π -Cartesian edge over Δ^1 , there is an extension $p(1,2) : \Delta^n \times \Delta^1\{1,2\} \rightarrow \mathcal{M}$ of $p(0,2)|_{\Delta^n \times \Delta^1\{2\}} \cup m|_{\partial\Delta^n \times \Delta^1\{1,2\}}$ covering σ by the dual of [20, Prop. 2.4.1.8].

Hence we obtain a map $q = m \cup p(1,2) \cup p(0,2) : (\partial\Delta^n \times \Delta^2) \cup (\Delta^n \times \Lambda_2^2) \rightarrow \mathcal{M}$ covering σ . We note that $q(\Delta^{\{i\}} \times \Delta^1\{1,2\})$ is π -Cartesian for all $i = 0, 1, \dots, n$.

Let \mathcal{E} be the set of edges of Δ^2 consisting of all degenerate edges together with $\Delta^1\{1,2\}$. We denote by $(\Delta^2)^+$ the marked simplicial set (Δ^2, \mathcal{E}) and by $(\Lambda_2^2)^+$ the marked simplicial set $(\Lambda_2^2, \mathcal{E} \cap (\Lambda_2^2)_1)$. The map of marked simplicial sets $(\Lambda_2^2)^+ \rightarrow (\Delta^2)^+$ is marked anodyne by [20, Def. 3.1.1.1]. This implies that $(\Delta^n)^b \times (\Lambda_2^2)^+ \cup (\partial\Delta^n)^b \times (\Delta^2)^+ \rightarrow (\Delta^n)^b \times (\Delta^2)^+$ is also marked anodyne by [20, Prop. 3.1.2.3].

Let $(\Delta^1)^\sharp$ be the marked simplicial set Δ^1 equipped with the set of all edges, and let \mathcal{M}^\sharp be the marked simplicial set \mathcal{M} equipped with the set of all π -Cartesian edges. Since $q(\Delta^{\{i\}} \times \Delta^1\{1,2\})$ is a π -Cartesian edge for all $i = 0, 1, \dots, n$, we have a map of marked simplicial sets $q : (\Delta^n)^b \times (\Lambda_2^2)^+ \cup (\partial\Delta^n)^b \times (\Delta^2)^+ \rightarrow \mathcal{M}^\sharp$. We consider the following commutative diagram of marked simplicial sets

$$\begin{array}{ccc} (\Delta^n)^b \times (\Lambda_2^2)^+ \cup (\partial\Delta^n)^b \times (\Delta^2)^+ & \xrightarrow{q} & \mathcal{M}^\sharp \\ \downarrow & \nearrow r & \downarrow \pi \\ (\Delta^n)^b \times (\Delta^2)^+ & \xrightarrow{\sigma} & (\Delta^1)^\sharp \end{array}$$

where the upper horizontal arrow is q . Since the left vertical arrow is marked anodyne, there is a dotted arrow r which makes the whole diagram commutative by [20, Prop. 3.1.1.6]. The proof is completed by setting $F = r|_{\Delta^n \times \Delta^1\{0,1\}}$ and $G = r|_{\Delta^n \times \Delta^1\{2\}}$.

□

References

1. A. Baker and A. Jeanneret, Brave new Hopf algebroids and extensions of MU -algebras, *Homology Homotopy Appl.* 4 (2002), no. 1, 163–173.
2. C. Barwick, S. Glasman and D. Nardin, Dualizing cartesian and cocartesian fibrations, preprint, 2014, arXiv:1409.2165.
3. A. K. Bousfield, The localization of spectra with respect to homology, *Topology* 18 (1979), no. 4, 257–281.
4. A. K. Bousfield and D. M. Kan, Homotopy limits, completions and localizations, *Lecture Notes in Mathematics*, Vol. 304. Springer-Verlag, Berlin-New York, 1972.
5. M. Behrens, and D. G. Davis, The homotopy fixed point spectra of profinite Galois extensions, *Trans. Amer. Math. Soc.* 362 (2010), no. 9, 4983–5042.
6. E. S. Devinatz and M. J. Hopkins, Homotopy fixed point spectra for closed subgroups of the Morava stabilizer groups, *Topology* 43 (2004), no. 1, 1–47.
7. D. G. Davis, Homotopy fixed points for $L_K(n)(E_n \wedge X)$ using the continuous action. *J. Pure Appl. Algebra* 206 (2006), no. 3, 322–354.
8. E. S. Devinatz, M. J. Hopkins, and J. H. Smith, Nilpotence and stable homotopy theory. I, *Ann. of Math. (2)* 128 (1988), no. 2, 207–241.
9. P. G. Goerss and M. J. Hopkins, Moduli spaces of commutative ring spectra, *Structured ring spectra*, 151–200, *London Math. Soc. Lecture Note Ser.*, 315, Cambridge Univ. Press, Cambridge, 2004.
10. K. Hess, A general framework for homotopic descent and codescent, preprint, 2010, arXiv:1001.1556.
11. M. J. Hopkins and J. H. Smith, Nilpotence and stable homotopy theory. II, *Ann. of Math. (2)* 148 (1998), no. 1, 1–49.
12. M. Hovey, Bousfield localization functors and Hopkins’ chromatic splitting conjecture, *The Čech centennial* (Boston, MA, 1993), 225–250, *Contemp. Math.*, 181, Amer. Math. Soc., Providence, RI, 1995.
13. M. Hovey, Morita theory for Hopf algebroids and presheaves of groupoids, *Amer. J. Math.* 124 (2002), no. 6, 1289–1318.
14. M. Hovey and H. Sadofsky, Invertible spectra in the $E(n)$ -local stable homotopy category, *J. London Math. Soc. (2)* 60 (1999), no. 1, 284–302.
15. M. Hovey, B. Shipley, and J. Smith, Symmetric spectra, *J. Amer. Math. Soc.* 13 (2000), no. 1, 149–208.
16. M. Hovey and N. P. Strickland, Morava K -theories and localisation, *Mem. Amer. Math. Soc.* 139 (1999), no. 666.
17. M. Hovey and N. Strickland, Comodules and Landweber exact homology theories, *Adv. Math.* 192 (2005), no. 2, 427–456.
18. P. S. Landweber, Homological properties of comodules over $MU_*(MU)$ and $BP_*(BP)$, *Amer. J. Math.* 98 (1976), no. 3, 591–610.
19. A. Lazarev, Towers of MU -algebras and the generalized Hopkins-Miller theorem, *Proc. London Math. Soc. (3)* 87 (2003), no. 2, 498–522.
20. J. Lurie, Higher topos theory, *Annals of Mathematics Studies*, 170. Princeton University Press, Princeton, NJ, 2009.
21. J. Lurie, Higher algebra (version 9/14/2014), available at <http://www.math.harvard.edu/~lurie/>.
22. J. Lurie, Derived algebraic geometry X: Formal moduli problems, preprint, 2011, available at <http://www.math.harvard.edu/~lurie/>.
23. A. Mathew, The Galois group of a stable homotopy theory, *Adv. Math.* 291 (2016), 403–541.
24. A. Mathew, N. Naumann, and J. Noel, Nilpotence and descent in equivariant stable homotopy theory, preprint, 2015, arXiv:1507.06869.
25. J. P. May, E_∞ ring spaces and E_∞ ring spectra, *Lecture Notes in Mathematics*, Vol. 577. Springer-Verlag, Berlin-New York, 1977.
26. H. R. Miller, D. C. Ravenel, and W. S. Wilson, Periodic phenomena in the Adams-Novikov spectral sequence, *Ann. of Math. (2)* 106 (1977), no. 3, 469–516.

27. J. Morava, Noetherian localisations of categories of cobordism comodules, *Ann. of Math.* (2) 121 (1985), no. 1, 1–39.
28. N. Naumann, The stack of formal groups in stable homotopy theory, *Adv. Math.* 215 (2007), no. 2, 569–600.
29. D. C. Ravenel, Complex cobordism and stable homotopy groups of spheres, *Pure and Applied Mathematics*, 121. Academic Press, Inc., Orlando, FL, 1986.
30. D. C. Ravenel, Nilpotence and periodicity in stable homotopy theory, *Annals of Mathematics Studies*, 128. Princeton University Press, Princeton, NJ, 1992.
31. T. Torii, Discrete G -Spectra and embeddings of module spectra, preprint, 2015, arXiv:1502.07800.